Lecture 1 / Week 1

Basics

We shortly review the basic consumer theory on preferences, since most of the models in asset pricing use the main assumptions of this theory. There is the set $\mathcal{X}$ which gives us what the consumer can choose and there is preference relation defined on that set. $(\mathcal{X}, \succeq)$ Most of the time in this class our set will look like $\mathcal{X} \equiv \mathbb{R}_{+}^{L}$.

Definition The **preference relation** $\succeq$ is defined as follows:

$$\begin{gathered}
\succeq \text{ preference relation: (at least as good as):} \\
\succ: x, y \in \mathcal{X} : x \succ y : x \succeq y \text{ but not } y \succeq x \text{ (strictly better)} \\
\sim: x, y \in \mathcal{X} : \ x \sim y : \ x \succeq y \text{ and } y \succeq x \text{ (indifference)}
\end{gathered}$$

We need to impose certain assumptions on preference relationship that we will assume to be true in this course unless stated otherwise:

1. $\succeq$ is **rational**: The rationality of preference relation is defined over completeness and transitivity:

$$\begin{aligned}
\text{completeness} \quad &: \quad x, y \in \mathcal{X} : x \succeq y \text{ or } y \succeq x \text{, or both.} \\
&\quad\ \ \text{(we should be able to compare all elements of our set)} \\
\text{transitivity} \quad &: \quad x, y, z \in \mathcal{X} : x \succeq y \ \text{ and } y \succeq z \Rightarrow \ x \succeq z. \\
&\quad\ \ \text{(we should be consistent in our ranking)}
\end{aligned}$$

2. $\succeq$ is **continuous**

   These two conditions together guarantee that we have a utility function s.t $\Rightarrow x \succeq y \Leftrightarrow u(x) \geq u(y)$: where $u(\mathbb{R}_{+}^{L}) \rightarrow \mathbb{R}$.

3. $\succeq$ is **monotone (strictly)**: This condition tells us the first derivative is increasing. Intuitively,it imposes that goods are desirable. (more is better!)

4. $\succeq$ is **convex (strictly):** We have concave indifference curves. Intuitively, it's good to mix the bundles, so agents appreciate diversity.

We also impose the additional mathematical condition on utility function that u(x) is twice continuously differentiable ($\mathcal{C}^2$). Under these conditions we have well behaved agents (concave utility functions) and finally we can deal with utility maximization.

$$\max_{x \geq 0} u(x)$$
$$s.t \;\; p^T(x-\omega) \leq 0 \text{ or}$$
$$p^T x \leq W = p^T \omega$$
ω is initial endowment and W is inital wealth.

Note that once we have the desirability condition, the budget constraint will be binding, so it will be an equality constraint. To solve this problem, we can set up the Lagrangian function. (In inequality case, we have to use Kuhn-Tucker conditions)

$$\begin{aligned} & \mathcal{L}(x,\lambda) = u(x) + \lambda(W - p^T x) \\ FOC: \quad & \tfrac{\partial u(x)}{\partial x_l} - \lambda p_l = 0, \; \forall \; l = 1, 2, ..L \\ & W - p^T x = 0 \end{aligned}$$

The above system caracterizes the solution: in vector form: $\nabla u(x^*) = \lambda$ p (Note that p is a vector with l components). So the gradient of the utility function at optimum, should be collinear with the prices vector. One can also see it graphically. (Orthogonality condition to budget constraint)

Insert here Figure 1

Economy

Definition The **economy** will be caracterized as a collection of choice sets, preferences and initial endowments over the agents.

$$\{\mathcal{X}_i, \succeq_i, \omega_i\}_i^I = \{\mathcal{X}_i, u()_i, \omega_i\}_i^I$$

Definition x=$\{x_1, x_2, ..., x_I\}$ is called an allocation. An allocation x is **feasible** iff:

$$\Sigma_{i=1}^I x_i \leq \Sigma_{i=1}^I \omega_i$$

In words, an allocation is feasible if and only if the aggregate consumption in the economy is not more than aggregate endowment. (Note: w.l.o.g, we have a pure exchange economy, so no production function.) Again, if we have the monotonicity (desirability condition) this inequality should hold with equality. $(\Sigma_{i=1}^I x_i = \Omega = \Sigma_{i=1}^I \omega_i)$

Definition A **Walrasian equilibrium** is caracterized by the optimal allocation and the optimum price vector: $(x^*\ , p^*)$. It is attained if the following two conditions are satisfied:

1. $(x^*\ , p^*)$ solves the UMP. (Utility Maximization Problem)

2. x^* is feasible. (As defined above)

So the Walrasian equilibruim says that in such an economy, all the agents should maximize their utility (Condition 1) and the markets should clear.(Condition 2) There is a quiete extensive literature on existence, uniqueness and properties of this equilibrium in general equilibrium literature, but in this course we will not focus on them.

Proposition Let $\{\mathcal{X}_i, u()_i, \omega_i\}_i^I$ be a "*standard economy*" and suppose that $\Sigma_{i=1}^I \omega_i \gg 0$. (positive endowment requirement). Then there exists a Walrasian equilibrium (x^* , p^*). We will not prove this proposition, but just use the results.

Definition A feasible allocation x is **Pareto optimal (efficient)** if there is no other allocation x$'$ (feasible) such that $u(x_i^{'}) \geq u(x_i) \; \forall \; i$ and $u(x_i^{'}) > u(x_i)$ for at least one agent. So in other words, there is no alternative way to allocate resources that makes some agents better off wihout making some other agent worse off. (Recall the example of voting with unanimaty).

FIRST WELFARE THEOREM

FWT: Let (x^* , p^*) be a Walrasian equilibrium for the economy $\{\mathcal{X}_i, u()_i, \omega_i\}_i^I$. Then the allocation x^* is Pareto efficient.

This theorem helps us to exploit the representative agent models. In such a set-up, the prices of single agent economy turn out to be the same as the prices in Walrasian equilibrium. We lose some important information on the distribution of consumption among agents, but in return we have the simplification through same prices.

Social Welfare Function: : W$(u_1(x_1), u_2(x_2), ...u_I(x_I)) = \frac{1}{I}\Sigma_{i=1}^I \sigma_i u_i(x_i)$, where $(\sigma_1, \sigma_2, ..., \sigma_I) > 0$. This social welfare function depends linearly(positive weights, σ_i) on the individual utilities of agents, where it is normalized by the number of agents. (per capita.) A Pareto optimum allocation can be implemented with a suitable choice of weights. The question is which choice of weights allow us to replicate the Walrasian equilibrium. The following two propositions give us the answer to this question.

Proposition Let x be the Pareto efficient allocation. Then x can be implemented through

$$\max_y \frac{1}{I}\Sigma_{i=1}^{I}\sigma_i u_i(x_i)$$
$$s.t \ \ \Sigma_{i=1}^{I}(z-y_i) \geq 0 \text{ (feasibility constraint)}$$
$$where\ z = \frac{\Omega}{I} \text{ (mean endowment)}$$

This is the so called **Social Planner Problem (SPP).** The social planner wants to maximize the social welfare function given the feasibility constraint, which tells us that the aggregate consumption cannot exceed aggregate endowment. Imposing the desirability condition will result in an equality constraint.

Recall the Walrasian equilibrium:

$$\nabla u(x_i^*) = \lambda_i\ p \quad \forall\ i = 1, 2...., I$$
$$p^* = \frac{\nabla u(x_1^*)}{\lambda_1} = \frac{\nabla u(x_2^*)}{\lambda_2} = ... = \frac{\nabla u(x_I^*)}{\lambda_I}$$

Note that all gradiants are constant across agents. This observation leads us to the following proposition.

Proposition Let $(x^*\ , p^*)$ be a Walrasian equilibrium and $\{\lambda_i\}_{i=1}^{I}$ be the Lagrange multiplier of the UMP. Then x^* can be implemented with $\sigma_i = \frac{1}{\lambda_i}\ \forall$ i in SPP.

Lecture 2 / Week 1

UNCERTAINTY

Conditionality Availability of a good depends on an exogenous event.(randomness).

Example Umbrella when it rains(London) and no rain(Barcelona) can be considered two different goods.

The concept of conditionality is modelled using **event trees**. It has 3 ingredients.

1. time: t: time period, t:0,1......T
2. $\mathcal{S}$ = The set of states of the world. s= single state, s:1,2,....S State: Rain / No rain.
3. Partition of $\mathcal{S}$: ϵ_t : a subset of $\mathcal{S}$ with some properties. These are events that can happen. ε_t in ϵ_t.

For the moment we have finite time in our models.

At t=0, there is complete uncertainty.

$\varepsilon_0 = \{1, 2, ...S\}$: **root**

$\varepsilon_T = \{\{1\}, \{2\}.....\{S\}\}$: uncertainty resolves sequentially. Over time we know what happens.

ε_o	ε_1	ε_2
We don't know	event1=$\{\{1,2,3\}\}$	State1
anything	event2=$\{\{4,5,6,7\}\}$	State2
	event3=$\{\{8,9\}\}$	.
	We have info that	
	we gather over time.	State9
info at t=0	t=1	t=2
$\mathcal{S}=3$		

2-Period Model

In this section we will focus on static portfolio choices and have only two periods: t: 0,1. In the above terminology, we have two partitions: $\varepsilon_0(root), \varepsilon_1$. In this models event trees are used to describe uncertainty. $\mathcal{S}=$ the state vector and we'll define L **contingent commodities**;i.e (comodity conditioned on the state(recall umbrella rain / no rain).

Definition For every physical commodity l=1,2.....L and state $\mathcal{S}=$ 0,1......S, a unit of contingent state commodity l$_s$ is a title to receive a physical comodity l if state s realizes.

Then we will have the following contigent comodity vector:

$\mathbf{x}=(x_1^0, x_2^0, \ldots.x_L^0, x_1^1, x_2^1, \ldots.x_L^1 \ldots\ldots x_L^S) \in \mathbb{R}^{(S+1)L}$.

The endowments can be defined the same way:

$\omega=(\omega_1^0, \omega_2^0, \ldots.\omega_L^0, \omega_1^1, \omega_2^1, \ldots.\omega_L^1 \ldots\ldots \omega_L^S) \in \mathbb{R}^{(S+1)L}$.

At time t$_0$: markets are open for trade of all state contingent commodities. We exchange payments(set prices), but the actual delivery only occurs if the state is realized,i.e. delivery depends on the state, not on the price. This set-up is similar to the one we had in the previous lecture, we can use Walrasian Equilibrium concept, but the difference is now we have more markets. ($(S+1)L$, instead of L markets. The problem the agent faces is the following:

$$\max_{x_i}\ u_i(x_i)$$

$$s.t\ \mathrm{p}^T\ (x_i-\omega_i) \leq 0$$

$$\text{where}\ \ \mathrm{p}^T_{(1x(S+1)L)}\ \ ,\ \ (x_i-\omega_i)_{(S+1)Lx\ 1}.$$

The solution would be exactly as before: $\nabla u_i(x_i^*)=\lambda p$

(a vector of $(S+1)L$ dimensions)

$$\frac{\frac{\partial u_i(x_i^*)}{\partial x_{l,i}}}{\frac{\partial u_i(x_i^*)}{\partial x_{k,i}}}=\frac{p_l}{p_k}$$

So, allocations only depend on relative prices. (*Classical dichotomy*)

Example t=1, W(wealth), d(damage) , s=1,2 (1= good state (W), 2= bad state(W-d)) . Agents can insure against damage paying a premium.

$\mu=$ full insurance premium

$c=$ coverage rate(0,1): How much of damage will be covered.

The agent has to decide on coverage rate.

$c\mu$ = insurance premium.

In a good state(bad state) agents will end up with W-$c\mu$; consumption at state 1 (W- d-$c\mu + cd$; consumption at State 2). The maximization problem of the agent becomes the following:

$$\max_c u(\text{W-}c\mu,\ \text{W- d-}c\mu + cd)$$
$$\text{FOC: -}\mu u_1 + u_2(d-\mu) = 0$$
$$\frac{u_1}{u_2} = \frac{d-\mu}{\mu}$$

where u_1 = the 1. derivative of utility w.r.t consumption at State 1.

$$\frac{\mu}{d} = \frac{p_2}{p_1+p_2}.$$

This is the basic result of consumption theory(marginal rate of substitution between two commodities equals the relative price of these two goods.): the premium per damage should be equal to the relative state contingent price of one unit of consumption on the bad state (relative to the price of risk-free(state independent) consumption.) As mentioned above the analyses of **Contingent Claim Economy** is the same as the standard economy, except the number of markets, hence the concepts like Walrasian equilibrium or Pareto efficient allocation can also be applied in this set-up.

Spot Markets

Now we will have a series of financial markets, namely spot markets. There finite number of states:

$$r^{j(asset)}_{S(state)} : cash\ flow, return$$

$$\text{r}^j = \begin{bmatrix} r^j_1 \\ r^j_2 \\ \cdot \\ \cdot \\ r^j_S \end{bmatrix}_{(Sx1)} \qquad \mathbf{r}= \begin{bmatrix} r^1_1 \ldots\ldots\ldots\ldots r^j_1 \\ \ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots \\ r^1_S \ldots\ldots\ldots\ldots r^j_S \end{bmatrix}_{(Sx\ J)} = \text{return matrix}$$

Example Risk-free return: independent of state: $\text{r}^{risk-free} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}_{(Sx1)}$

The important question is in which unit the cash flows should be defined. (Numeraire). There are some conventions like average consumption(a basket) or the first good in the vector.(Mas -Colell). So, defining return depends on the choice of numaraire.

Definition Arrow Security: $e^S = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}_{(Sx1)}$. This security only provides in State S one unit of purchasing power of the numeraire.

$$\mathbf{e}= \begin{bmatrix} 1.............0 \\ 0.1...........0 \\ ....1.......... \\ ....1...... \\ 0...............1 \end{bmatrix}_{(Sx\ S)}$$

q_j = Price of asset j

$\mathbf{q}=(q_1, q_2............q_J)$

α_S = Price of Arrow security

$\alpha =(\alpha_1, \alpha_2............\alpha_S)$

Definition A **portfolio** is a collection of asset units bought and sold:

$$\mathbf{z}=\begin{bmatrix} z_1 \\ z_2 \\ . \\ . \\ z_j \end{bmatrix}_{(Jx1)} .$$

Definition **Return on a portfolio** (cash-flow) is given by

$$\mathbf{r}_{(Sx\ J)}\mathbf{z}_{(Jx1)} = \begin{bmatrix} \sum_{j=1}^{J} r_1^j z_j \\ \sum_{j=1}^{J} r_2^j z_j \\ . \\ \sum_{j=1}^{J} r_S^j z_j \end{bmatrix}_{(Sx1)}$$

Definiiton **Cost of portfolio** is given by $\mathbf{q}_{(1xJ)}\mathbf{z}_{(Jx1)}$. (Note that it is a scalar.)

Law of One Price

If we have two portfolios $\mathbf{z}_{(Jx1)}$ and $\mathbf{z}'_{(Jx1)}$, then law of one price postulates if $\mathbf{r}_{(Sx\ J)}\mathbf{z}_{(Jx1)} = \mathbf{r}_{(Sx\ J)}\mathbf{z}'_{(Jx1)} \Rightarrow \mathbf{q}_{(1xJ)}\mathbf{z}_{(Jx1)} = \mathbf{q}_{(1xJ)}\mathbf{z}'_{(Jx1)}$. In words, two assets with the same payoff vector have same prices.

An application of law of one price is Arrow securities: $\mathrm{r}^j_{(Sx1)} = e_{(Sx\ S)} r^j_{(Sx\ 1)}$. In words, a portfolio on Arrow security gives the same return as the original asset. Then the cost of the portfolio: $\mathrm{q}_j = \alpha_{(1xS)} * r^j_{(Sx1)}.(Decomposition)$ So, we can replicate assets using Arrow securities.

Example A financial asset that pays 1 in state 1, 3 in state 2 and 0 in state 3 has the same state contingent payoff as a portfolio of 1 state 1, 3 state 2 and 0 state 3 Arrow securities. $\mathrm{r}^j = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1\ 0\ 0 \\ 0\ 1\ 0 \\ 0\ 0\ 1 \end{bmatrix} * \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$

Risk-Neutral Probabilities

The risk neutral probability is defined as $\widetilde{\alpha}_s = \frac{\alpha_s}{\beta}$, where β is the price of risk free asset. Note that these are not probabilities in its strict sense,.i.e assigning likelihood to the states, but the components of it sum up to1.

$$\beta = \alpha_{(1xS)} r^0_{(Sx1)} = \sum_{s \in S} \alpha_s \ (r^0 = \text{risk-free asset})$$

Definition $\widetilde{E}$ is defined as the **expectation operator** under risk neutral probabilities.

Then the price of an asset j is $\mathrm{q}_j = \beta\widetilde{E}(r^j)_{(Sx1)} = \frac{\widetilde{E}(r^j)}{\rho}$, where ρ(risk free return: $\frac{1}{price} = \frac{1}{\beta}$). In arbitrage pricing theory, the aim is to find risk neutral probabilities to price assets. Note that the expected gross rate of return of ant asset, evaluated with risk-neutral probabilities , equals the risk-free rate of return. $\Rightarrow \widetilde{E}(R^j) = \rho$, where $R^j_s := \frac{r^j_s}{q_j}$.

The asset economy is defined as $\{(X_i, u_i, \omega_i)^I_{i=1}$, $\mathbf{r}$ $\}$

Market Span

Market span is the set of possible asset portfolio(caracterized by the cost and return) and denoted by $\mathcal{M}$(q)= span$\begin{bmatrix} -q \\ r \end{bmatrix}$ $=: \left\{ \begin{bmatrix} -q \\ r \end{bmatrix}.z \mid z \in \mathbb{R}^J \right\}$

We also define $\alpha_+ = (1\ \alpha_1\ \alpha_2\ ....\ \ \alpha_s)$, note that it is same as α defined before with a leading 1. This vector is orthogonal to $\mathcal{M}$(q).

Proof Consider an element of market span; x $\in \mathcal{M}$(q). $\Rightarrow x = \begin{bmatrix} -q \\ r \end{bmatrix}.z$ for some portfolio z. But then, $\alpha_+ \begin{bmatrix} -q \\ r \end{bmatrix}.z = 0$, since (-q+$\alpha$r) =0 (Law of one price).

Decision Problem

The agent maximizes utility by choosing today's consumption (x^0) , possible consumption bundles that may be realized tomorrow ($\mathrm{x}^1, ...\mathrm{x}^S$) and a portfolio of securities to satisfy the budget constraint at every t and state s. This decision problem can be called the integrated consumption portfolio problem. Note that the future spot prices ($\mathrm{p}_1, ...\mathrm{p}_S$) are not observable today, but the agents have beliefs about future spot prices. (B(p_1),......B(p_S)). These beliefs are conditional on state and not subject to uncertainty.

$\max_{x,z}\ u_i(x)$
s.t $\mathrm{p}_0^T\ (x_i^0 - \omega_i^0) + q.z \leq 0$,where [$\mathrm{p}^T_{0_{(1xL)}}\ (x_i^0 - \omega_i^0)_{(Lx1)}$]= $-$ saving, $q.z =$ investment (scalar)
B(p_s)$_{(Lx1)}$*$(x^s - \omega^s)_{(Lx1)}$-$\mathrm{r}_{s_{(1xJ)}}.z_{(Jx1)} \leq 0$ for s=1,2....S, where
[B(p_s)$^T_{(1xL)}.(x^s - \omega^s)_{(Lx1)}$]= the value of excess consumption,
$\mathrm{r}_{s_{(1xJ)}}.z_{(Jx1)} = return.(* : inner\ product)$.

The problem here is that the solution might not exist. We know that the solution exists when the objective function is continuous on a compact(closed and bounded set). The utility function is continuous by construction, by assumption the budget set is closed, but boundedness might create problem(monotonic utility, more is better)$\Rightarrow$ Arbitrage argument. (See example P.48)

Definition (q,r) contains **arbitrage opportunities** if there exists a portfolio z, s.t

$$\begin{bmatrix} -q \\ r \end{bmatrix} .z \geq 0$$

This means that this portfolio gives non-negative cash-flows today or tomorrow and it is strictly positive either today or in at least one future state. Formally, the absence of arbitrage oppotunities can be expressed as follows: the market span must not intersect with the positive orphant except the origin.

$$\mathcal{M}(q) \cap \mathbb{R}_+^{S+1} = \{0\}$$

We have proved that α_+ is orthogonal to $\mathcal{M}(q)$.

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Lecture 3 / Week 2

Arbitrage

Example Assume that $S = 1, J = 1$ and there is only risk free asset with return r^0 and price β. Then (q,r) arbitrage condition becomes $\begin{bmatrix} -\beta \\ 1 \end{bmatrix} z \geq 0$. There is arbitrage either if $\beta = 0$ or if $\beta < 0$. This can be seen graphically in this special case. ($\mathcal{M}(q) \cap \mathbb{R}_+^{S+1} = \{0\}$) Note that this intersection never becomes empty since we are always able to choose z=0.

Insert here Figure 1

Proposition Arbitrage opportunities are absent ((q,r) arbitrage free) if and only if there exists an $\alpha \gg 0$ (strictly positive Arrow prices, risk neutral probabilities) s.t $q_{(1xL)}=\alpha_{(1xS)}.r_{(SxL)}$

Proof Let y=$\begin{bmatrix} -q \\ r \end{bmatrix} z \geq 0$ be arbitrage opportunity. Let $\alpha \gg 0$, then $\alpha_+ = [1 \ \alpha] \gg 0$, and hence $\alpha_+ y > 0$, but this contradicts with orthogonality condition we have proved; α_+ is orthogonal to $\mathcal{M}(q)$, in other words the multiplication of any vector of market span with α_+ should be 0. Conversely, if the market allows for arbitrage, then the arbitrage cash flow (the market span as a whole) cannot be orthogonal to any strictly positive α_+.

Relating this proposition to the previous example, since we had only one asset, namely the risk free asset, no arbitrage condition is only consistent with positive price β.

Radner Equilibrium

Let **z**$= (z_i)_{i=1}^{I}$ be the collection of portfolios for individuals in the economy. Then the **market clearing condition for financial assets** becomes

$$\sum_{i=1}^{I} z_i = 0$$

since for each financial asset there should exist one buyer and one seller.

Let $\{\mathrm{B}^i(p)\}_{i=1}^{I}$ be the beliefs on prices of state contingent goods tomorrow. Beliefs satisfy **conditional perfect foresight (CPF)** if

$$p_s = B^i(p_s) \text{ for } \forall i \in I, \forall s \in S$$

This condition tells us that equilibrium requires that every agent in the economy should have the same belief on the prices and these beliefs should be correct. However, this foresight is conditional on state, and there is still uncertainty about the state, so unconditional perfect foresight is not assumed.

The market clearing condition(**MCC**) on physical goods and the utility maximization problem (**UPM)** remains the same, namely

1. **MCC on physical goods**: $\sum_{i=1}^{I} x_i^s = \sum_{i=1}^{I} \omega_i^s$, for $\forall s \in S$. (Also known as **feasibility** constraint)
2. **UPM**: $\max_{x_i,z_i} u_i(x)$

   s.t $p_0^T\ (x_i^0 - \omega_i^0) + q.z \leq 0$

   $B^i(p_s)_{(Lx1)} * (x_i^s - \omega_i^s)_{(Lx1)} - r_{s_{(1xJ)}}.z_{(Jx1)} \leq 0$ for s=1,2....S

Definition Let $\{(X_i, u_i, \omega_i)_{i=1}^{I}, \mathbf{r}\ \}$ be the *asset economy.*

The 4-tuple $(x^*_{((S+1)xL)xI}, p^*_{((S+1)xL)}, z^*_{(Jx1)}, q^*_{(1xJ)})$ is called **Radner Equilibrium**, if it fulfills the following conditions;

1. x^* solves **UPM.**
2. The **MCC**(feasibility constraint) holds**.**
3. z^* fulfills the market clearing condition for financial assets.
4. Beliefs satisfy the conditional perfect foresight. (CPF)

The problem can be split into two parts, namely **consumption composition problem(CCP)** and **financial problem(FP)**.

First notice that under monotonicity assumption and applying CPF, the budget constraint for UMP can be written with equality as follows

$$\begin{aligned} p_0 * (x_i^0 - \omega_i^0) &= -q.z \\ p_{s(Lx1)} * (x_i^s - \omega_i^s)_{(Lx1)} &= r_{s_{(1xJ)}}.z_{(Jx1)} \end{aligned}$$

Writing the budget constraint explicitly in vector form

$$\begin{pmatrix} p_0 * (x_i^0 - \omega_i^0) \\ p_1 * (x_i^1 - \omega_i^1) \\ . \\ . \\ p_S * (x_i^S - \omega_i^S) \end{pmatrix}_{(S+1)x1} = \begin{bmatrix} -q \\ r \end{bmatrix} z$$

In short notation the decision problem becomes

$$\max_{x_i,z_i} \{u_i(x_i) |\ p\ *\ (x_i - \omega_i) \in \mathcal{M}(q)\}$$

CCP

$$\max_{x_i}\ u_i(x)$$
$$\text{s.t } \mathrm{p}_s x_i^s \leq y^s,\ \text{s=0,1,2 } ...S$$

where y=$(y^0, y^1, ...y^S)$ is the income distribution (today and tomorrow), $\mathrm{p}_s x_i^s$ = money spent on consumption.

The indirect utility is defined as follows

$$\begin{aligned} v(y) &= u_i(\widehat{x}_i) \\ & \quad where\ \ \widehat{x}_i \text{ is the optimum consumption.} \\ v(y) &: = \{u(x)|\ \mathrm{p}_s x^s \leq y^s \text{ for s=0,1,...} S\} \\ & \quad v(y) \text{ is the maximized utility if at most y}^s \text{ can be spent in state } s. \end{aligned}$$

FP

$$\max_{z_i} v(y)$$
$$\text{s.t } \mathrm{p}_0\omega_i^0 - qz_i = y^0$$
$$\mathrm{p}_s\omega_i^s + r_s z_i = y^s$$

where $\mathrm{p}_s\omega_i^s$ is the value of endowment and qz_i is the cost of the portfolio.

We define $\mathrm{W}_i^s = \mathrm{p}_s\omega_i^s$,s=0,1,2 ...$S$

then the budget constraint becomes

$$y^0 - W_i^0 = -qz_i$$
$$y^s - W_i^s = r_s z_i$$

Then the problem can be written as

$$\max_{z_i} v(y)$$
$$\text{s.t } y - W_i \in \mathcal{M}(q)$$

Even shortly

$$\max_{z_i}\{v(y)|\ y - W_i \in \mathcal{M}(q)\} = \max_{z_i}\{\max_{x_i}\{u_i(x_i)|\ p_s x_i^s = y^s,\ \forall s\ |$$
$$y - W_i \in \mathcal{M}(q)\ \} =$$
$$= \max_{x_i, z_i}\{u_i(x_i)|\ p * (x_i - \omega_i) \in \mathcal{M}(q)\}.$$

By splitting the problem into two; CCP and FP, and introducing indirect utility function $v(y)$ and state contingent value of endowment W_i^s, we can consider a new economy, 2-tuple(v,W), a contingent claim economy with only one commodity, consumption (income) in today and each future state. By using a representative commodity, we lose information on composition of consumption, but still caracterize equilibrium asset prices.

Reverse Decomposition

We have seen as an application of Law of one price, that a general asset can be respresented by a portfolio of Arrow securities. (Decomposition:$\mathrm{q}_j = \alpha_{(1xS)} * r^j_{(Sx1)}$). Is the converse true? Can we generate an Arrow security with a combination of general financial assets? I.o.w is **reverse decomposition** possible?

The answer is it depends! In general we cannot say that we can always do reverse decomposition; it depends on the size and structure of the return matrix **r.**

Example If there are 5 states and only one financial asset, we cannot generate $\alpha_1, \alpha_2.., \alpha_5$ simply by observing the price $q_1.$

Reverse decomposition is possible if and only if financial assets provide diverse enough state-contingent cash-flows. ($J \geq S$)

Definition Markets are **complete** if agents can insure each state seperately.

If markets are complete then there is a portfolio for each state that generates the state contingent cash flows of the state-s Arrow security. Formally, for each state s there exists a portfolio z^s s.t

$$\begin{aligned} r_{(SxJ)}.z^s_{(Jx1)} &= e^s, \ \forall s \in S \\ r.[z^1......z^S] &= \mathbf{e} \end{aligned}$$

The portfolios z^s can only be computed(reverse decomposition) iff **r** is *convertible*

$$[z^1......z^S] = r^{-1}.$$

So, markets are complete iff **r** is convertible. Then Arrow prices can be computed from the financial market prices: $\alpha = q.r^{-1}$.(**Uniqueness** of Arrow prices) Mathematically, this condition to hold, the rank of the return matrix should be equal the number of states. ($J \geq S \Longleftrightarrow rank(\mathbf{r}) = S$). If this is not the case, there are many possible Arrow prices that are compatible with return matrix **r** and financial market prices q.

Lecture 4 / Week 2

LOTTERY

Gambles (risky situations) can be represented by a list of possible outcomes (pay-off, return) and their respective probabilities. Formally;

$$L = [x_1, \pi_1; x_2, \pi_2.........x_S, \pi_S], \ \pi_s \geq 0 \text{ and } \sum_{s=1}^{S} \pi_s = 1$$

This is called a Lottery (L), where x, π, s represent the payoff, probability and states, respectively. Why do we need lotteries? Because they will help us to represent risky assets. Some special types of lotteries are:

Binary Lottery: $[x_1, \pi; x_2, 1 - \pi]$

Degenerate Lottery: $[x, 1]$.

Compound Lottery defined on L_1, L_2: $L_c = [L_1, \pi; L_2, 1 - \pi] \Rightarrow$ *we* can construct the reduced lottery.

Let the set of all lotteries be $\mathcal{L}$. We will define preferences over this set like in ordinal utility theory with the usual assumptions. (asymetric, negatively transitive, continuous) Then we can represent such preferences with a continuous function $\upsilon : \mathcal{L} \to \mathbb{R}$. s.t $L \succeq L^{'} \Leftrightarrow \upsilon(L) \succeq \upsilon(L^{'})$.

$$\upsilon : \mathcal{L} \to \mathbb{R}.s.t\ L \succeq L^{'} \Leftrightarrow \upsilon(L) \succeq \upsilon(L^{'}).$$

We will have three main assumptions to have expected utility representation over lotteries, namely,

1. **consequentialism**: People are indifferent between reduced and compound lotteries. $\Rightarrow L^R \sim L^C$
2. **state independence**: $[x, \pi; y, 1-\pi] \sim [y, 1-\pi; x, \pi]$. This assumption tells us it should be the price(payoff) not the label that affects the decision.
3. **irrelavance of common alternatives**: Assume we have $L, L^{'}, L^{"} \in \mathcal{L}$ and $\alpha \in (0,1)$, then

$$L \succeq L^{'} \Leftrightarrow\ \alpha L + (1-\alpha)L^{"} \succeq \alpha L^{'} + (1-\alpha)L^{"}$$

So in words, the common alternatives should not change the direction of our prefence relation. Under these assumptions with the assumptions we made on preferences we can define the **expected utility representation(EUR)**:

$$U(L) = U([x_1, \pi_{1;}x_2, \pi_2.........x_S, \pi_S]) = \sum_{s=1}^{S} \pi_s u(x_s)$$

where $u(x_s)$ is the von Neumann-Morgenstern utility. ($U(L) = E(u(L))$. In other words, von Neumann-Morgenstern utility represent EUR as a linear function of probabilities. Notice that EUR is an **ordinal** utility function, i.e. any monotonic transformation does not change the ranking thus represent the same preferences, whereas $u(x_s)$ is an **cardinal** utility function that is invariant only under positive affine transformations. (i.e $\widetilde{u} \equiv u \Leftrightarrow \exists a, \exists b > 0$,$\forall x$, $\widetilde{u}(x) = a + bu(x)$).

RISK

Definition Let L be a non-degenerate Lottery and $L^{'}$ be a degenerate lottery with $L^{'} = [E(L), 1]$. Then we can classify risk attitutes of investors as follows;

i) investor is **risk-averse**:= $U(L^{'}) > U(L)$ $\quad U(E(L)^{'}) > U(L)$

ii) investor is **risk-neutral**:= $U(L^{'}) = U(L)$ $\quad U(E(L)^{'}) = U(L)$

iii) investor is **risk-lover**:= $U(L^{'}) < U(L)$ $\quad U(E(L)^{'}) < U(L)$

Insert here Figure 1(4.3,P.74, L(2004))

In this figure we have shown three lotteries: $L := [z, 1]$ $z > 0$, $L^{'} := [\frac{z}{\pi_1}, \pi_1; 0, \pi_2]$,$L^{"} := [0, \pi_1; \frac{z}{\pi_2}, \pi_2]$ and showed that we have a similar curves like a budget constraint and indifference curves($p_1x_1 + p_2x_2 = w$ vs $\pi_1x_1 + \pi_2x_2 = z$)and that the gradient of the utility vector is collinear with the probabilities.

Definition The amount of money that an investor willing to pay to be indifferent between taking the sure amount of money and the lottery is called certainty equivalent amount. This phenomenon is called **certainty equivalence**. Formally, $U([CE(L), 1]) = U(L)$. Another description of risk aversion is, an investor is risk averse iff $E(L) > CE(L)$. The difference $RP(L) = E(L) - CE(L)$ is called **risk premium**. Note that positive risk premium⇔ risk aversion.

Risk aversion can also be shown graphically ⇒**concave** von Neumann-Morgenstern utility function⇔ risk aversion. ($U(E(L)) \geq E(u(L))$)Formally, $|u^{"}_A(w)|>|u^{"}_B(w)|$, for $\forall w$, then investor A is more risk averse than investor B.

Insert here Figure 1(4.8,P.80, L(2004))

We have already mentioned that $u_A(x)$ is invariant only under positive affine transformations. (i.e $\widetilde{u} \equiv u \Leftrightarrow \exists a, \exists b > 0\ , \forall x,\ \widetilde{u_A}(x) = a + bu_A(x))$, but than $\widetilde{u}_A^{''}(w) = bu_A^{''}(w), b > 0$. So, concavity alone (second derivative of utility funciton) is not a proper measure for risk, since it gives different levels for the same investor. We'll normalize it by the first derivative and define:

$$\begin{aligned} ARA(Absolute\ Risk\ Aversion) &= A_u(w) = -\frac{u^{''}(w)}{u^{'}(w)} \\ RRA(\mathrm{Re}\,lative\ Risk\ Aversion) &= w.A_u(w) = R_u(w) \end{aligned}$$

These measures are invariant under any affine transformations. $(A_u(w) = A_{\widetilde{u}}(w), R_u(w) = R_{\widetilde{u}}(w))$. To see the usefullness of these measures we'll define the following binary lottery: $[h, \pi; -h, 1-\pi]$ and the probability $\widetilde{\pi} = \pi(h, w)$, i.e the probability that makes the investor take the lottery or not. We also define

$$u(w) = \widetilde{\pi}.u(w+h) + (1-\widetilde{\pi}).u(w-h) \qquad (*)$$

where LHS= the utility by refusing the lottery, RHS= the expected utility of accepting the lottery. Then

$$\widetilde{\pi} \simeq \frac{1}{2} + \frac{1}{4}.h.A_u(w)$$

Proof We take the following Taylor approximations around h=0 (P.60-61, DD 2005);

$$\begin{aligned} u(w+h) &= u(w) + h.u^{'}(w) + \frac{h^2}{2}.u^{''}(w) + o(h^2) \\ u(w-h) &= u(w) - h.u^{'}(w) + \frac{h^2}{2}.u^{''}(w) + o(h^2) \end{aligned}$$

We subsitiute these approximations in (*) and solve for $\widetilde{\pi}$ using $A_u(w)$ definition. This $\widetilde{\pi}$ tells us that more risk averse(i.t.o ARA) people require higher probabilities for positive return to accept the lottery.

In a similar fashion, it can also be shown for RRA, where we define $h = \theta.w$ (payoff relative to wealth) and the lottery $L = [\theta.w, \pi; (1-\theta.w), 1-\pi]$ and approximate a la Taylar around $\theta = 0$, we show that

$$\widetilde{\widetilde{\pi}} \simeq \frac{1}{2} + \frac{1}{4}.\theta.R_u(w)$$

Definition We say that the utility function exhibits constant absoulute risk aversion, CARA, if $A_u^{'}(w) = 0$,i.e. A does not depend on wealth, so the same measure would hold ,independent how rich the person is. (IARA= increasing absoulute risk aversion, DARA=decreasing absoulute risk aversion.)

$$\begin{aligned} CARA &: A_u^{'}(w) = 0 \\ IARA &: A_u^{'}(w) > 0 \\ DARA &: A_u^{'}(w) < 0 \end{aligned}$$

Definition We say that the utility function exhibits constant relative risk aversion, CRRA, if $R_u^{'}(w) = 0$,i.e. R does not depend on wealth, so the same measure(in relative terms to initial wealth) would hold ,independent how rich the person is. (IRRA= increasing relative risk aversion, DARA=decreasing relative risk aversion.)

$$\begin{aligned} CRRA &: R_u^{'}(w) = 0 \\ IRRA &: R_u^{'}(w) > 0 \\ DRRA &: R_u^{'}(w) < 0 \end{aligned}$$

Empirical evidence shows that most agents have 1.) strictly increasing, 2.) strictly concave and 3.) DARA utility functions with not too large relative risk aversion. (0<R(w)<4 $\forall w$).

Lecture 5 / Week 3

STOCHASTIC DOMINANCE

Instead of evaluating the risky projects just by mean-variance analysis, a general approach has been developed using the probability distributions. We define $F_L(x) :=$ **cumulative distribution function.** Then we can compare two Lotteries L and $L^{'}$ in the following way;

Definition Let $F_L(x)$ and $F_{L'}(x)$ be defined over [0,1]. Then $F_L(x)$ **FSD (first order stochastically dominates)** $F_{L'}(x)$ iff $F_L(x) \leq F_{L'}(x)$.

Insert here Figure 1(4.5,P.68, DD(2005))

Proposition $F_L(x)$ **FSD** $F_{L'}(x) \Leftrightarrow E(u(L)) - E(u(L^{'})) \geq 0$ with u nondecreasing.

Proof

"$\Leftarrow$ " : (Proof by Contradiction, $\neg b \rightarrow \neg a$) Assume that $\exists\ \overline{x} \in$[0,1] and $F_L(x) - F_{L'}(x) > 0$.($Not\ L$ FSD $L^{'}$). We define the following nondecreasing utility function $\overline{u}(x)$: (MC notation: $U(F) = \int u(x)dF(x)$ where $U() = vNM$, $u() = Bernoulli$)

$$\overline{u}(x) = \begin{Bmatrix} 1 & \overline{x} \leq x \leq 1 \\ 0 & 0 \leq x \leq \overline{x} \end{Bmatrix}$$

$$
\begin{aligned}
Then E(\overline{u}(L)) - E(\overline{u}(L^{'})) &= \int_0^1 \overline{u}(x) dF_L(x) - \int_0^1 \overline{u}(x) dF_{L'}(x). \\
& (Expected\ value\ represented\ by\ integrals.) \\
&= \int_0^1 \overline{u}(x) d(F_L(x) - F_{L'}(x)) (Since 0 \leq x \leq \overline{x} \Rightarrow \overline{u}(x) = 0) \\
&= \int_{\overline{x}}^1 \overline{u}(x) d(F_L(x) - F_{L'}(x)) \\
Then\ we\ integrate\ by\ parts &: Recall \int udv = [u.v] - \int vdu \\
&= [\overline{u}(x).(F_L(x) - F_{L'}(x))]_{\overline{x}}^1 - \int_{\overline{x}}^1 (F_L(x) - F_{L'}(x)).\overline{u}^{'}(x) dx \\
(Notice\ that\ F_L(1) &= F_{L'}(1) = 1\ and\ \overline{u}^{'}(x) = 0 (constant\ function)) \\
&= -(F_L(\overline{x}) - F_{L'}(\overline{x})) < 0. \\
(Since\ by\ assumption\ F_L(x) - F_{L'}(x) &> 0, then\ we\ reach\ contradiction. \Rightarrow E(\overline{u}(L)) - E(\overline{u}(L^{'})) < 0)
\end{aligned}
$$

"$\Rightarrow$" : W.l.o.g: Assume a differentiable $u(x)$ with $u^{'}(x) > 0$.
Note again that

$$E(\mathrm{u}(L)) - E(\mathrm{u}(L^{'})) = \int_0^1 u(x)dF_L(x) - \int_0^1 u(x)dF_{L'}(x)$$

$$= \int_0^1 u(x)d(F_L(x) - F_{L'}(x))$$

$$(Integration\ by\ parts)$$
$$= [u(x).(F_L(x) - F_{L'}(x))]_0^1 - \int_0^1 (F_L(x) - F_{L'}(x)).u^{'}(x)dx$$
$$(First\ term = 0\ as\ before)$$

$$= \int_0^1 (F_{L'}(x) - F_L(x)).u^{'}(x)dx \geq 0.$$
$$(We\ used\ FSD\ and\ the\ fact\ u^{'}(x)\ positive.QED.)$$

Insert here Figure 1

Note that it is not state by state dominance, but still extremely strong condition. Since it does not use the concept of risk aversion, one can hope for a broader measure that includes risk aversion.

Definition Let $F_L(x)$ and $F_{L'}(x)$ be two cumulative distribution functions defined over [0,1], and we have two lotteries with the same expected value, i.e. $\int_0^1 x.dF_L(x) = \int_0^1 x.dF_{L'}(x)$.Then $F_L(x)$ **SSD (second order stochastically dominates)** $F_{L'}(x)$ iff $\int_0^x (F_L(t) - F_{L'}(x))dt \leq 0$ $\forall x \in$ [0,1].

Insert here Figure 2

Notice that from the graph in case FSD the curves of cumulative distribution never cross, but this might be the case in case of SSD. Also, note that the area between curves are the same.

Proposition $F_L(x)$ **SSD** $F_{L'}(x)$ (L is less risky than $L^{'}$) iff $E(u(L)) > E(u(L^{'}))$ for u "concave, (risk averse agent)" ($u^{'} \geq 0, u^{''} < 0$).
Proof Similar to the previos one, but slightly more complicated.

Insert here Figure 3

Definition Let $f_L(x)$ and $f_{L'}(x)$ be the probability density functions of two lotteries. If $f_{L'}(x)$ can be obtained from $f_L(x)$ by removing some of the probability weight from the center of $f_L(x)$ and distributing it to the tails in such a way that the mean is unchanged, than $f_{L'}(x)$ is related to $f_L(x)$ via a **mean preserving spread**. In other words, the two lotteries have the same expected return, but $L^{'}$ is riskier than L. (higher variance.)

STATIC FINANCE ECONOMY

- We define the time seperable vNM utility in the following way: u(y^0)+ $\delta.u(y)$, where δ represents the parameter for time preference(impatience). How much we value todays consumption over future consumption. Usually, $\delta \in (0,1)$, meaning that we prefer to consume now rather than later.
- We have the same states for all assets in the economy.

Before defining our problem, we should note that we make the following simplifications:

1. The expected utility is calculated in the following way: $E(u(y)) = \sum_{s=1}^{S} \pi_s.u(y^s)$
2. Today agents are provided with wealth w^0, but tomorrow there is no wealth provided in any of the states: $w^s = 0$, $s = 1, 2, ...S$.
3. We have two assets in the economy: the risk-free asset, r^0, and a risky asset r.

Then we can define the **canonical portfolio problem** in the following way:

$$\begin{aligned} & \max_{z_0,z} E(u(y)) \\ s.t\ q.\mathbf{z}(q^0.z^0 + q.z) \quad &\leq \quad w^0 \\ y^s \quad &\leq \quad w^s + r^s.\mathbf{z},\ \forall s \end{aligned}$$

Similary,in terms of value assigned to stocks,

$$\begin{aligned} \max_{z_0,z^1} \sum_{s=1}^{S} \pi_s.u(y^s) \quad &= \quad U(y^1, y^2....y^S) \\ s.t\ (q^0.z^0 + q.z^1) \quad &\leq \quad w^0 \\ y^s \quad &\leq \quad r^0.z^0 + r_1^s.z^1,\ \forall s = 1, 2.....S \end{aligned}$$

Then we can define values in terms of today's prices:

$$\begin{aligned} \widetilde{z}^i &= q_i.z^i \quad i = 0, 1 \\ \widetilde{r}^0 &= \frac{r^0}{q^0} \\ \widetilde{r}^s &= \frac{r^s}{q_1} \end{aligned}$$

Following we describe the same problem in values with today's prices(i.t.o tildas) assuming monotonicity;

$$\begin{aligned}
& \max_{z_0,z^1}\sum_{s=1}^{S}\pi_s.u(y^s) \\
z^0+z^1 &= w^0 \Rightarrow z^0 = w^0 - z^1 \\
y^s &= r^0.z^0 + r_1^s.z^1 \Rightarrow y^s = r^0.(w^0-z^1)+r_1^s.z^1 \\
&= r^0.w^0 + z^1(r_1^s - r^0) \\
where\ (r_1^s - r^0) &= risk\ premium
\end{aligned}$$

Then the portfolio choice can be presented just in terms of money invested in risky asset:

$$\begin{aligned}
\max_{z^1}\sum_{s=1}^{S}\pi_s.u(r^0.w^0+z^1(r_1^s-r^0)) &= E(u(y)) \\
FOC &: E[u'(y^*).(r_1^s-r^0)] = 0 \\
&= \sum_{s=1}^{S}\pi_s.u'(y_s^*).(r_1^s-r^0)
\end{aligned}$$

Proposition $z_1^* > 0$ (optimum amount invested in risky asset)$\Leftrightarrow E(r_1) > r^0$

$$\begin{aligned}
z_1^* &> 0 \Leftrightarrow E(r_1) > r^0 \\
z_1^* &= 0 \Leftrightarrow E(r_1) = r^0 \\
z_1^* &< 0 \Leftrightarrow E(r_1) < r^0
\end{aligned}$$

Proof Define the function $f(z) = E[u'(y).(r_1^s - r^0)]$, take the first derivative: $f^{'}(z) = E[u^{''}(y).(r_1^s - r^0)^2]$ (negative slope: $(r_1^s - r^0)^2 > 0$ and $u^{''}(y) < 0$). We also have that $f(z^*) = 0 \Leftrightarrow u'(y^*) = 0$. Then

$$\begin{aligned}
z^* &= 0(\text{No risky investment} \Rightarrow f(0) = E[u'(r^0.w^0).(r_1^s-r^0)] \\
& \quad \sin ce\ (r^0.w^0) \text{ is not stochastic} \\
&= u^{'}(r^0.w^0).E[(r_1^s-r^0)] \\
u^{'}(r^0.w^0) &> 0 \Leftrightarrow E[(r_1^s-r^0) = 0 \Leftrightarrow E(r_1^s) = r^0.
\end{aligned}$$

$$\begin{aligned}
z^* &> 0 \Leftrightarrow f(0) > 0 \\
u'(y) &> 0 \Rightarrow E[(r_1^s-r^0) > 0 \Leftrightarrow E(r_1^s) > r^0 \\
z^* &< 0 \Leftrightarrow f(0) < 0 \\
u'(y) &> 0 \Rightarrow E[(r_1^s-r^0) < 0 \Leftrightarrow E(r_1^s) < r^0. \text{ QED.}
\end{aligned}$$

Insert here Figure 4

Lecture 6 / Week 3

Proposition $z_1^* > 0$ (optimum amount invested in risky asset)$\Leftrightarrow E(r_1) > r^0$

$$\begin{aligned} z_1^* &> 0 \Leftrightarrow E(r_1) > r^0 \\ z_1^* &= 0 \Leftrightarrow E(r_1) = r^0 \\ z_1^* &< 0 \Leftrightarrow E(r_1) < r^0 \end{aligned}$$

Recall that to prove the previous proposition we used the following function $f(z) = E[u'(y).(r_1 - r^0)]$ it takes value 0 at optimum investment, hence

$$E[u'(y^*).(r_1 - r^0)] = 0 \text{ , where } y^s = w^0.r^0 + z.(r_1^s - r^0) \quad s = 1, 2.....S$$

$$y^s = w^0.(r^0 + \tfrac{z}{w^0}.(r_1^s - r^0))$$

$$y^s = w^0.r_p^s$$

$$\tfrac{z}{w^0} = weight \text{ i.t.o ratio relative to initial wealth}$$

$$(r^0 + \tfrac{z}{w^0}.(r_1^s - r^0)) = r_p^s \text{ gross return on portfolio.}$$

Proposition The optimum investment on risky asset for a "**small risk**" can be approximated by

$$\begin{aligned} z^* &\simeq \frac{E(r_1 - r^0)}{v(r_1 - r^0).A(w^0.r^0)} \\ E(r_1 - r^0) &= \text{expected excess return} \\ v(r_1 - r^0) &= \text{variance of excess return} \\ A(w^0.r^0) &= ARA \text{ coefficient.} \end{aligned}$$

Before proceding to the proof we can state the intuition. Keep in mind that this explanation holds for small risks. The higher the risk the worst is the approximation relative to the true value. This ratio tells us, the higher excess return is expected over the risk-free rate, the more will be the agent investing on risk asset. By the same token, the higher the expected variance of the excess return, and the more risk averse is the agent, the less she will be willing to invest on risky asset.

Proof We make a Taylor approximation of the marginal utility around the risk-free wealth $(w^0.r^0)$, i.e if the agent had invested all her wealth on risk-free asset.

$$\begin{aligned}
u^{'}(y^s) &= u^{'}(w^0.r_p^s) \\
u^{'}(w^0.r_p^s) &\simeq u^{'}(w^0.r^0) + u^{''}(w^0.r^0).z.(r_1^s - r^0) + o(r_1^s - r^0) \quad /*.(r_1^s - r^0) \\
u^{'}(w^0.r_p^s).(r_1^s - r^0) &\simeq u^{'}(w^0.r^0).(r_1^s - r^0) + u^{''}(w^0.r^0).z.(r_1^s - r^0)^2 + o[(r_1^s - r^0)^2] \ \ /*\text{Exp} \\
E[u^{'}(w^0.r_p).(r_1 - r^0)] &\simeq u^{'}(w^0.r^0).E[(r_1 - r^0)] + u^{''}(w^0.r^0).z.E[(r_1 - r^0)^2] + o[E(r_1 - r^0)^2] \\
\text{using the fact } E[u'(y^*).(r_1 - r^0)] &= 0,\ E[(r_1 - r^0)^2] = v(r_1 - r^0) \text{ and } o[E(r_1 - r^0)^2] \text{ negligible} \\
0 &\simeq u^{'}(w^0.r^0).E[(r_1 - r^0)] + u^{''}(w^0.r^0).z^*.v(r_1 - r^0) \\
z^* &\simeq -\frac{u^{'}(w^0.r^0).E[(r_1 - r^0)]}{u''(w^0.r^0).v(r_1 - r^0)} \\
\sin ce\ A(w^0.r^0) &\simeq -\frac{u^{''}(w^0.r^0)}{u^{'}(w^0.r^0)} \\
z^* &\simeq \frac{E(r_1 - r^0)}{v(r_1 - r^0).A(w^0.r^0)}. \quad \text{QED.}
\end{aligned}$$

Example

Assume we have a logarithmic utility function $u(w) = \ln(w)$. We have risky(r_1) and one risk-less asset (r^0). Tomorrow we will have a return (r_1^1) on risky asset with probability π and return (r_2^1) with probability $1 - \pi$. Since we have a risk averse agent by assumption, $E(r_1) > r^0$. We will also have the following "*no arbitrage condition*". $r_1^1 > r^0 > r_1^2$. This inequality must hold, otherwise the agent exploit the return to make arbitrage. (Buying cheap and (short) selling high).

We have the following optimization problem

$$\begin{aligned}
& \max_{z^0,\ z^1}\ \pi.\ln(y^1)+(1-\pi).\ln(y^2) \\
s.t\ z^0+z_1 &= w^0 \Leftrightarrow z^0=w^0-z_1 \\
y^s &= z^0.r^0+z_1.r_1^s \quad \text{s=1,2.....S} \\
y^s &= (\mathbf{w}^0-\mathbf{z}_1).r^0+z_1.r_1^s \\
y^s &= w^0.r^0+z_1.(r_1^s-r^0) \\
& \text{ the problem becomes only a decison of risky investment} \\
& \max_{z^1}\ \pi.\ln(y^1)+(1-\pi).\ln(y^2) \\
s.t\ y^s &= w^0.r^0+z_1.(r_1^s-r^0) \\
FOC\ &:\ \frac{\pi}{y^1}.(r_1^1-r^0)+\frac{1-\pi}{y^2}.(r_1^2-r^0)=0 \Longleftrightarrow E[u'(y^*).(r_1-r^0)]=0 \\
\pi.y^2.(r_1^1-r^0) &= (1-\pi).y^1.(r^0-r_1^2) \\
y^2 &= (w^0.r^0)+z_1(r_1^2-r^0),\ \ y^1=(w^0.r^0)+z_1(r_1^1-r^0) \\
\pi.((w^0.r^0)+z_1(r_1^2-r^0)).(r_1^1-r^0) &= (1-\pi).((w^0.r^0)+z_1(r_1^1-r^0)).(r^0-r_1^2) \\
(w^0.r^0).[\pi.(r_1^1-r^0)-(1-\pi)(r^0-r_1^2)] &= 2\pi.z_1.(r_1^1-r^0)).(r^0-r_1^2) \\
\text{where } [\pi.(r_1^1-r^0)-(1-\pi)(r^0-r_1^2)] &= \pi.r_1^1+(1-\pi).r_1^2-r^0, \quad \boldsymbol{\pi}.\mathbf{r}_1^1+(\mathbf{1}-\boldsymbol{\pi}).\mathbf{r}_1^2=\mathbf{E}(\mathbf{r}) \\
(w^0.r^0).(E(r)-r^0) &= 2\pi.z_1.(r_1^1-r^0).(r^0-r_1^2) \\
z^* &= \frac{w^0.r^0.(E(r)-r^0)}{(r_1^1-r^0).(r^0-r_1^2)}
\end{aligned}$$

The important point to note in this example is that we have a linear relationship between the investment in risky asset and initial wealth. In other words the fraction of the initial wealth invested in risky asset does not depend on initial wealth, hence we have CRRA, i.e constant relative risk aversion. This comes from the logarithmic utility function.

One might want to ask to question of how big should the expected excess return be so that the risk averse agent is induced to make risky investment. The following inequality provides the answer:

Proposition

$$E(r) - r^0 \geq w^0.A(w^0.r^0).v(r_1 - r^0)$$

This is the lower bound that induces the agent to invest all her money on risky asset.

Proof The proof is almost the same as before, we just put the restriction is that all wealth should be invested in risky asset and no short-selling constaint.($z \geq 0$)

$$\begin{array}{rcl} u^{'}(y^s) & = & u^{'}(w^0.r_p^s) \\ \text{optimality condition} & : & E[u'(y^*).(r_1 - r^0)] = 0 \\ & & E[u'(w^0.r^0 + z_1(r_1 - r^0)).(r_1 - r^0)] \\ \text{whole initial wealth} & : & E[u'(w^0.r^0 + w^0(r_1 - r^0)).(r_1 - r^0)] \\ z & = & w^0 \geq 0 \text{ (no shortselling constraint)} \\ u^{'}(w^0.r_p^s) & \simeq & u^{'}(w^0.r^0) + u^{''}(w^0.r^0).z.(r_1^s - r^0) + o(r_1^s - r^0) \\ u^{'}(w^0.r_p^s).(r_1^s - r^0) & \simeq & u^{'}(w^0.r^0).(r_1^s - r^0) + u^{''}(w^0.r^0).z.(r_1^s - r^0)^2 + o[(r_1^s - r^0)^2] \\ E[u^{'}(w^0.r_p).(r_1 - r^0)] & \simeq & u^{'}(w^0.r^0).E[(r_1 - r^0)] + u^{''}(w^0.r^0).z.E[(r_1 - r^0)^2] + o[E(r_1 - r^0)^2] \\ 0 & \simeq & u^{'}(w^0.r^0).E[(r_1 - r^0)] + u^{''}(w^0.r^0).z^*.v(r_1 - r^0) \\ E(r) - r^0 & \geq & w^0.A(w^0.r^0).v(r_1 - r^0) \end{array}$$

Proposition Higher expected return on risky investment is equivalent to saying negative covariance with the return of risky asset and marginal utility evaluated at the optimal portfolio.

$$\begin{array}{rcl} E(r_1) & \geq & r^0 \Leftrightarrow cov(r_1, u^{'}(w^0.r_p^*)) \leq 0 \\ E(r_1) & \leq & r^0 \Leftrightarrow cov(r_1, u^{'}(w^0.r_p^*)) \geq 0 \end{array}$$

Proof We will prove the first line, the proof is analogous for the second line. We have the following FOC for the UMP

$$E[u'(w^0.r_p^*).(r_1 - r^0)] = 0$$

Recall the formula:

$$cov(X, Y) = E(X.Y) - E(X).E(Y) \Leftrightarrow E(X.Y) = cov(X, Y) + E(X).E(Y)$$

Then we can apply this formula to the optimality condition

$$\begin{array}{rcl} E[u'(w^0.r_p^*).(r_1 - r^0)] & = & cov(u'(w^0.r_p^*), (r_1 - r^0)) + E(u'(w^0.r_p^*)).E((r_1 - r^0)) = 0 \\ & = & cov(u'(w^0.r_p^*), r_1) + E(u'(w^0.r_p^*)).E((r_1 - r^0))) = 0 \\ cov(u'(w^0.r_p^*), r_1) & = & -E(u'(w^0.r_p^*)).E((r_1 - r^0))) \end{array}$$

Since $u'(w^0.r_p^*) > 0$ by assumption of monotonicity, the terms $cov(u'(w^0.r_p^*), r_1)$ and $E((r_1 - r^0)$.have opposite signs. This observation completes the proof. QED.

Notice that negative covariance with return of risky asset and marginal utility implies that there is a positive covariance between the return of risky asset and wealth at $\mathrm{t}_1(w^0.r_p^*)$. In other words, the investors demand positive risk premium for those assets that provide bad insurance. Reversing the argument, the investors might prefer risky assets over the riskless one, i.e. willing to pay a risk premium, if it provides an insurance once a bad state occurs. This refers to the second equality.

The above proposition holds only under two cases:

1. **Normal distribution**: If the return of the risky asset is normally distributed. ($\mathrm{r}_1 \sim N$)

   Proof We have to show that if the return of the risky asset is normally disributed, then $E(r_1) \geq r^0 \Leftrightarrow cov(r_1, w^0.r_p^*) \geq 0$. We will use the following lemma in the proof;

   Stein's Lemma: Let X, Y be bivariate normal random variables. Let $g : \mathbb{R} \rightarrow \mathbb{R}$ so that $E(|g(Y)|) < \infty$. Then

   $$cov(g(Y).X) = E[g^{'}(Y)].cov(X, Y)$$

   We set $Y = w^0.r_p^*$, $X = r_1$ (Note that since $(r^0 + \frac{z}{w^0}.(r_1^s - r^0)) = r_p^s \Rightarrow r_1 \sim N \Rightarrow r_p \sim N$) and $g(Y) = u^{'}(w^0.r_p^*)$. Using the lemma

   $$cov(u^{'}(w^0.r_p^*), r_1) = E[u^{''}(w^0.r_p^*)].cov(w^0.r_p^*, r_1)$$

   Since $u^{''}(w^0.r_p^*) < 0$, by risk aversity assumption, $cov(u^{'}(w^0.r_p^*), r_1)$ and $cov(w^0.r_p^*, r_1)$ have opposite signs and this observation combined with the previous proof completes the proof. QED.

2. **Quadratic Utility**: If the agent has quadratic utility as

   $$\begin{aligned} u(w) &= \gamma_0 w - \gamma_1 w^2 \\ u^{'}(w) &= \gamma_0 - 2\gamma_1 w > 0 \\ u^{''}(w) &= -2\gamma_1 < 0, \ \gamma_1 > 0 \\ cov(r_1, \gamma_0 - 2\gamma_1 w^0.r_p^*) &= cov(r_1, -2\gamma_1 w^0.r_p^*) \\ &\quad -cov(r_1, \gamma_1 w^0.r_p^*) \end{aligned}$$

We can see that $cov(u^{'}(w^0.r_p^*), r_1)$ and $cov(w^0.r_p^*, r_1)$ have opposite signs and this observation completes the proof. QED.

Lecture 7 / Week 4

We will conduct the following analysis, how does the demand for risky asset changes with initial wealth? Do people make more investment on risk assets once they become richer? This questions can be summarized formally,

$$z^{'}(w^0) = \frac{dz^*}{dw^0}$$

Proposition Suppose that $z^*(w^0) > 0 \Leftrightarrow E(r_1) \geq r^0$. (Risk averse agent). Then

$$\begin{aligned} A^{'}(w) &< 0 \text{ (DARA)} \Leftrightarrow z^{'}(w^0) > 0 \\ A^{'}(w) &= 0 \text{ (CARA)} \Leftrightarrow z^{'}(w^0) = 0 \\ A^{'}(w) &> 0 \text{ (IARA)} \Leftrightarrow z^{'}(w^0) < 0 \end{aligned}$$

The first case tells us that the absolute risk aversion decreases with increasing wealth. In other words, the more richer one gets the more demands the risky asset.

Proof We will define the following function $f(z) = E[u^{'}(w^0.r_p).(r_1 - r^0)]$. We know from the shape of the utility function (concave) that $f^{'}(z) < 0 \Leftrightarrow u" < 0$. We also know from the optimality condition to UMP, that $f(z^*) = 0$. Since we want to make the analysis $z^{'}(w^0) = \frac{dz^*}{dw^0}$, we will use the implicit function theorem

$$\frac{dz^*}{dw^0} = -\frac{\frac{\partial f(z^*)}{\partial w^0}}{\frac{\partial f(z^*)}{\partial z^*}}$$

But since $\frac{\partial f(z^*)}{\partial z^*} < 0$, we know that $sign(\frac{dz^*}{dw^0}) = sign(\frac{\partial f(z^*)}{\partial w^0})$. So we will use RHS to make conclusion about LHS of the equation. Recall that $(r^0 + \frac{z}{w^0}.(r_1^s - r^0)) = r_p^s \ s = 1, 2...S$, so $w^0.r_p^s = (w^0.r^0 + z.(r_1^s - r^0)) \ s = 1, 2...S$. Then we calculate the following derivative

$$\frac{\partial f}{\partial w^0} = E[u^{"}(w^0.r_p).(r_1 - r^0).r_p] \quad (*)$$

We need to show that $A^{'}(w) < 0 \Leftrightarrow z^{'}(w^0) > 0$.

We will compare the marginal utility at two different wealth levels, namely, $w^0.r^0$ vs. $w^0.r_p^s$, we know by assumption that $z > 0$. There are two cases for comparison

$$\begin{aligned} Case\ 1 &: r_1^s \geq r^0 \Rightarrow w^0.r_p^s \geq w^0.r^0 \Rightarrow u'(w^0.r_p^s) \leq u^{'}(w^0.r^0) \\ Case\ 2 &: r_1^s < r^0 \Rightarrow w^0.r_p^s < w^0.r^0 \Rightarrow u'(w^0.r_p^s) > u^{'}(w^0.r^0) \end{aligned}$$

By the same token, we conduct a similar analysis, since $A^{'}(w) < 0$

$$\begin{aligned} Case\ 1 &: r_1^s \geq r^0 \Rightarrow w^0.r_p^s \geq w^0.r^0 \Rightarrow A(w^0.r_p^s) \leq A(w^0.r^0) \\ Case\ 2 &: r_1^s < r^0 \Rightarrow w^0.r_p^s < w^0.r^0 \Rightarrow A(w^0.r_p^s) > A(w^0.r^0) \end{aligned}$$

We multiply both sides with $u'(w^0.r_p^s) > 0$ and $(r_1^s - r^0)$.

$Case\ 1 : r_1^s \geq r^0 \Rightarrow (r_1^s - r^0) > 0 \Rightarrow u'(w^0.r_p^s).(r_1^s - r^0).A(w^0.r_p^s) \leq u'(w^0.r_p^s).(r_1^s - r^0)A(w^0.r^0)$

Using the definition of $A(w^0.r_p^s) = -\frac{u^{"}(w^0.r_p^s)}{u'(w^0.r_p^s)}$

$$-u^{"}(w^0.r_p^s).(r_1^s - r^0) \leq u'(w^0.r_p^s).(r_1^s - r^0)A(w^0.r^0)$$

$Case\ 2 : r_1^s < r^0 \Rightarrow (r_1^s - r^0) < 0 \Rightarrow u'(w^0.r_p^s).(r_1^s - r^0).A(w^0.r_p^s) < u'(w^0.r_p^s).(r_1^s - r^0).A(w^0.r^0)$.

$$-u^{"}(w^0.r_p^s).(r_1^s - r^0) < u'(w^0.r_p^s).(r_1^s - r^0)A(w^0.r^0)$$

Then we take the expectation, sum up across states

$$\begin{aligned} E[-u^{"}(w^0.r_p).(r_1 - r^0)] &< E[u'(w^0.r_p).(r_1 - r^0)A(w^0.r^0)] \\ E[-u^{"}(w^0.r_p).(r_1 - r^0)] &< A(w^0.r^0).E[u'(w^0.r_p).(r_1 - r^0)] \end{aligned}$$

Once we evaluate at the optimum $E[u'(w^0.r_p^*).(r_1 - r^0)] = 0$. Hence

$$\begin{aligned} E[-u^{"}(w^0.r_p).(r_1 - r^0)] &< 0 \\ E[u^{"}(w^0.r_p).(r_1 - r^0)] &> 0. \end{aligned}$$

This completes the proof, once we see the relation to (*). Since by assumption r$_p > 0$, then $E[u^{"}(w^0.r_p).(r_1 - r^0)] \Leftrightarrow \frac{\partial f}{\partial w^0} > 0 \Leftrightarrow A^{'}(w) < 0$. QED.

We might also want to answer the question of the weight: $\frac{amount\ \text{invested in risky asset}}{initial\ \text{wealth}}$ changes once the initial wealth changes. This question is related to the relative risk aversion. In other words, what is the percentage change in investment in risky asset once the initial wealth changes in percentage terms. (elasticity). Does people invest a higher fraction of their wealth in risky asset once they become richer? Formally we conduct the following analysis

$$\eta^* = \frac{dz^*}{dw^0}.\frac{w^0}{z^*}$$

Proposition Let $R(w)$ be the **relative risk aversion** coefficient, then

$$\begin{aligned} R^{'}(w) &< 0 \text{ (DRRA)} \Leftrightarrow \eta(w^0) > 1 \Leftrightarrow \text{the risky asset is normal good} \\ R^{'}(w) &= 0 \text{ (CRRA)} \Leftrightarrow \eta(w^0) = 1 \\ R^{'}(w) &> 0 \text{ (IRRA)} \Leftrightarrow \eta(w^0) < 1 \Leftrightarrow \text{ the risky asset is inferior good} \end{aligned}$$

We will not proof this proposition, but the interpretation is important; if your wealth changes **in percentage terms**, DRRA implies that the agent wants to invest more in risky asset, so in other words the risky asset is a normal good. (Analogly, in the last case, it is inferior good.) This analysis is the most we can conduct with the canonical portfolio problem. From now on, we will analyse cases, where we have more than one risky asset. Then, we will need another framework to see the relationship between the risk premia of different risky assets and its effect on portfolio choice.

Example The following function is DARA (CRRA)

$$\begin{aligned}
u(w) &= \frac{1}{1-\gamma}w^{1-\gamma} \\
u^{'}(w) &= w^{-\gamma} \\
u''(w) &= -\gamma.w^{-(\gamma+1)} \\
A(w) &= -\frac{u^{''}(w)}{u(w)} = \frac{\gamma.w^{-(\gamma+1)}}{w^{-\gamma}} = \frac{\gamma}{w} \\
A^{'}(w) &= -\frac{\gamma}{w^2} < 0 \ (DARA) \\
R(w) &= \gamma \text{ (CRRA)}
\end{aligned}$$

Lecture 8 / Week 4

Modern Portfolio Theory (Mean -Variance Analysis)

We will see that mean-variance analysis can be successfully conducted in three cases.

1. **Quadratic utility**.
2. **Jointly normal distribution of portfolio returns**.
3. **Approximation in the neighbourhood of** $E(X)$.

Case 1 Quadratic Utility: We have the following utility function : $u(w) = \gamma_0.w - \gamma_1.w^2$. Now there are n assets in the portfolio and we consider the following lottery:

$$x = [x_1, \pi_1; ....x_s, \pi_s]$$

The expected utility becomes

$$E[u(x)] = \sum_{s=1}^{S} \pi_s.u(x_s)$$

We substitute our quadratic utility

$$\begin{aligned} E[u(x)] &= \sum_{s=1}^{S} \pi_s.(\gamma_0.x_s - \gamma_1.x_s^2) = \\ &= \gamma_0.\sum_{s=1}^{S} \pi_s.x_s - \gamma_1.\sum_{s=1}^{S} \pi_s.(x_s^2) = \\ &= \gamma_0.E(X) - \gamma_1.var(X) - \gamma_1.(E(X))^2 \end{aligned}$$

Recall that $var(X) = E(X^2) - (E(X))^2 \Leftrightarrow E(X^2) = var(X) + (E(X))^2$.
To see how the expected utility changes:

$$\begin{aligned} \frac{d(E(u(x)))}{d(E(X))} &= \gamma_0 - 2\gamma_1.E(X) \Longleftrightarrow u^{'} = \gamma_0 - 2\gamma_1 w > 0 \\ \frac{d(E(u(x)))}{d(var(X))} &= -\gamma_1 < 0 \end{aligned}$$

We see that $\uparrow E(X) \Rightarrow \uparrow E[u(x)]$ and $\uparrow \sigma(X) \Rightarrow \downarrow E[u(x)]$, so

$$\begin{aligned} E[u(x)] &= f(E(X), \sigma(X)). \\ E[u(x)] &= f(\uparrow, \downarrow) \end{aligned}$$

Insert here Figure 1

Note that the quadratic utility is IARA and is not empirically justified (DARA is common.)

Case 2 Jointly normal distributed returns: We assume that we have jointly normal returns $X \sim N(\mu, \sigma^2)$. We can standardise the returns $X = E(X) + \sigma(X).z$, where $z \sim N(0,1)$. Then we can write the expected value of the utility as follows

$$\begin{aligned} E[u(X)] &= E[u(E(X)) + \sigma(X).z] \\ \frac{d(E(u(X)))}{d(E(X))} &= E[u^{'}(X)] > 0 \\ \frac{d(E(u(X)))}{d(\sigma(X))} &= E[u^{'}(X).z] < 0 \end{aligned}$$

The last equality follows from both the concavity of the utility function and by the fact that z is symmetric. (negative components are bigger.)

Insert here Figure 2

Limited liability nature of financial instruments makes in general the normality assumption on returns empirically unjustified, but if we compound continously, then the cumulative continuously compounded returns are normally disributed. ($r^c = \log(1+r)$, r^c is lognormally distributed) Empirical evidence says, daily stock returns are skewed to the right, certain stock indeces are skewed to the left and many individual stocks' daily return distribution exhibit excess kurtosis(fat tails as well as stock indices, in monthly data less kurtosis is observed. Another complication might arise, since the lognormality of individual stocks does not imply the lognormality of portfolio returns (log of a sum is not equal to the sum of the log), but if we take short periods(daily) the errors are usually small.)

Case 3 Mean Variance Analysis as Approximation: We assume that $u(X)$ is well defined, i.e. it has finite moments in the neighbourhood of E(X). Then we can have the following Taylor expansion:

$$\begin{aligned}
u(X) &\simeq u(E(X)) + u^{'}(E(X)).(X - E(X)) + \frac{1}{2}.u^{"}(E(X)).(X - E(X))^2 \\
We\ take\ \exp ectation &: \\
E(u(X)) &\simeq u(E(X)) + u'(E(X)).(E(X) - E(X)) + \frac{1}{2}.u^{"}(E(X)).\sigma^2(X) \\
\frac{d(E(u(X)))}{d(E(X))} &= u^{'}(E(X)) + \frac{1}{2}.u^{'''}(E(X)).\sigma^2(X) \quad > 0 \\
u^{'''} &: curvature\ of\ m\arg inal\ utility,\ u^{'''} > 0 \rightarrow sufficient\ condition \\
\frac{d(E(u(X)))}{d(\sigma(X))} &= \frac{1}{2}.u^{"}(E(X)).2.\sigma(X) \quad < 0
\end{aligned}$$

Note that once we have quadratic utility $u^{'''} = 0$, then the condition is automatically satisfied.

Insert here Figure 3

Reduced Form of Preferences

Since we know that the expected utility is a inreasing function in expected value of returns and a decreasing one in variance of returns we will use the following simplified utility function to conduct mean variance analysis

$$E(X) - \alpha\sigma^2(X),\ \alpha > 0$$

We hope to get closed form solution for asset demands. We want to be able to write the budget constraint and to rank the investment opportunities according to some criterion, which will be the following

Definition Assuming that the agent has investment opportunities with the same expected return, i.e. $E(X_1) = E(X_2)$, the first one dominates the second one i.t.o mean variance, i.e. X_1 **mean variance dominates** X_2 if $\sigma(X_1) < \sigma(X_2)$. This is called **mean-variance criterion. (M-V criterion)**

Our next step will be to find the budget constraint that satisfies the M-V criterion, once the agent has a portfolio with n assets. Then

$$q_1.z_1 + q_2.z_2 + ...q_n.z_n = w^0$$

We divide by initial wealth since our focus in our analysis is about portfolio choice, i.e. how the agent split its wealth among different portfolio assets. (Before we did the analysis for the change w.r.t. initial wealth.)

$$\frac{q_1.z_1}{w^0} + ...\frac{q_n.z_n}{w^0} = 1$$

Each term in the summation is called **portfolio weight** that are normalized by initial wealth. We change the notation to $w_1 = \frac{q_1.z_1}{w^0}$. Then

$$w_1 + w_2 + ...w_n = 1$$

is socalled the fully invested portfolio. Then the net portfolio return will be

$$r_p = w_1.r_1 + w_2.r_2....... + w_n.r_n$$

Matrix Notation

portfolio weights: $\mathbf{w}=\begin{bmatrix} w_1 \\ w_2 \\ . \\ . \\ w_n \end{bmatrix}_{n\times 1}$ $\quad$ *var-cov matrix* : $\mathbf{v}=\begin{bmatrix} \sigma_1^2 & \sigma_{12} & . & \sigma_{1n} \\ & \sigma_2^2 & & \\ & & . & \\ & & & \sigma_n^2 \end{bmatrix}_{n\times n}$

exp*ected return* : $\boldsymbol{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ . \\ . \\ \mu_n \end{bmatrix}_{n\times 1}$ $\quad$ *unit vector* $\mathbf{i}=\begin{bmatrix} 1 \\ 1 \\ . \\ . \\ 1 \end{bmatrix}_{n\times 1}$

Portfolio Choice

Case 1 n risky assets, no risk-free rate

Before writing the optimization problem we will express our terms in matrix form:

$$\begin{aligned}
\mu_p &: = E(r_p) = w_1.\mu_1 + w_2.\mu_2.....w_n.\mu_n = \boxed{\mathbf{w}^{'}.\boldsymbol{\mu}}_{(1\times n)\times(n\times 1)} \\
\sigma_p^2 &: = var(r_p) = \boxed{\mathbf{w}^{'}.\mathbf{v}.\mathbf{w}}_{(1\times n)\times(n\times n)\times(n\times 1)} \\
\textbf{Budget constraint} &: \mathbf{w}^{'}.\mathbf{i} = 1
\end{aligned}$$

Then the optimization problem becomes

$$\begin{aligned}
&\min_{\mathbf{w}} \mathbf{w}^{'}.\mathbf{v}.\mathbf{w} \\
s.t\ \mathbf{w}^{'}.\boldsymbol{\mu} &= \mu_T \text{ (target return)} \\
\mathbf{w}^{'}.\mathbf{i} &= 1 \text{ (Budget constraint)}
\end{aligned}$$

So we have constructed an optimization problem i.t.o. mean variance criterion. The agent wants to reach a target return that minimizes the variance of its portfolio and satisfies at the same time the budget constraint.

Example Two risky asset case:

$$\begin{aligned}
\min_{\mathbf{w}} \sigma_p^2 &= w_1^2.\sigma_1^2 + w_2^2.\sigma_2^2 + 2.w_1.w_2.\sigma_{12} \\
s.t\ \ w_1.\mu_1 + w_2.\mu_2 &= \mu_T \\
w_1 + w_2 &= 1 \Leftrightarrow w_2 = 1 - w_1
\end{aligned}$$

Then we can write the optimization problem only with one control variable

$$\begin{aligned}
\min_{w_1} \sigma_p^2 &= w_1^2.\sigma_1^2 + (1-w_1)^2.\sigma_2^2 + 2.w_1.(1-w_1).\varphi.\sigma_1.\sigma_2 \\
where\ cov(r_1, r_2) &= \varphi.\sigma_1.\sigma_2 = \sigma_{12} \\
s.t\ \ w_1.\mu_1 + (1-w_1).\mu_2 &= \mu_T
\end{aligned}$$

We will analyse special cases

Ex. Case 1 $\varphi = 1$: perfect positive correlation between returns of two risky asset , then we have

$$\begin{aligned}
\sigma_p^2 &= [w_1.\sigma_1 + (1-w_1).\sigma_2]^2 \Leftrightarrow \sigma_p = w_1(\sigma_1 - \sigma_2) + \sigma_2 \Leftrightarrow w_1 = \frac{\sigma_p - \sigma_2}{\sigma_1 - \sigma_2} \\
\mu_T &= w_1.\mu_1 + (1-w_1).\mu_2 \Leftrightarrow \mu_T = w_1.(\mu_1 - \mu_2) + \mu_2 = (\frac{\sigma_p - \sigma_2}{\sigma_1 - \sigma_2}).(\mu_1 - \mu_2) + \mu_2
\end{aligned}$$

If we want to see the relationship between portfolio standard deviation σ_p (x-axis) and the portfolio return μ_p (y-axis) graphically, we obtain a straight line. This curve, which is a line in this special case will be called **portfolio**

frontier. This shows us the linear relationship between the return and risk (i.t.o standard deviation) of the portfolio in this special case.

Ex. Case 2 $\varphi = -1$: perfect negative correlation between returns of two risky asset , then we have

$$\sigma_p = \left\{ \begin{array}{c} w_1.\sigma_1 - (1-w_1).\sigma_2 \\ -w_1.\sigma_1 + (1-w_1).\sigma_2 \end{array} \right\}$$

Then we have two line one with positive and one with negative slope intersecting at y-axis.(**risk-free portfolio**). Then the the portfolios on the line with with positive slope(portfolio frontier) are more effcient (M-V dominate) than the ones on the negative line.

Between these two extreme case, i.e. $-1 < \varphi < 1$, we will have a curve that lies between these two extreme cases, where we can find the minimum variance portfolio. (Only the second special case it is 0.)

Insert here Figure 4

Now we will proceed with how to solve such an optimization problem with n risky asset without a riskless one. We will set-up the Lagrangian and make use of matrix algebra.

Recall the problem

$$\begin{aligned} & \min_{\mathbf{w}} \frac{1}{2}\mathbf{w}^{'}.\mathbf{v}.\mathbf{w} \\ s.t\ \mathbf{w}^{'}.\boldsymbol{\mu} &= \mu_T \\ \mathbf{w}^{'}.\mathbf{i} &= 1 \end{aligned}$$

Note that $\frac{1}{2}$ just brings computational simplicity and does not change the nature of the optimization problem. Recall also from matrix algebra

$$\begin{aligned} \frac{d(a^{'}.b)}{da} &= b \\ \frac{d(a^{'}.A.a)}{da} &= 2.A.a \end{aligned}$$

Then

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}\mathbf{w}^{'}.\mathbf{v}.\mathbf{w}+\lambda.(\mu_T - \mathbf{w}^{'}.\boldsymbol{\mu})+\gamma.(1-\mathbf{w}^{'}.\mathbf{i}) \\ FOC &: \frac{\partial\mathcal{L}}{\partial\mathbf{w}} = \frac{1}{2}.2.\mathbf{v}.\mathbf{w}-\lambda.\boldsymbol{\mu}-\gamma.\mathbf{i} = \mathbf{0} \qquad (n\times 1) \\ \frac{\partial\mathcal{L}}{\partial\boldsymbol{\lambda}} &= \mu_T - \mathbf{w}^{'}.\boldsymbol{\mu} = 0 \\ \frac{\partial\mathcal{L}}{\partial\gamma} &= 1 - \mathbf{w}^{'}.\mathbf{i} = 0 \end{aligned}$$

Using matrix algebra and the fact that **v** is invertible (square matrix), we can write n+2 optimality conditions in the following way:

$$\begin{aligned}
\mathbf{w} &= \mathbf{v}^{-1}.(\lambda.\boldsymbol{\mu}{+}\gamma.\mathbf{i}) \quad \text{(n conditions)} \\
& \text{we premultiply with } \boldsymbol{\mu}' \\
\boldsymbol{\mu}'.\mathbf{w} &= \boldsymbol{\mu}'.\mathbf{v}^{-1}.(\lambda.\boldsymbol{\mu}{+}\gamma.\mathbf{i}) = \mu_T \\
& \sin ce\ \lambda \text{ and } \gamma \text{ are scalars} \\
\boldsymbol{\mu}'.\mathbf{w} &= (\boldsymbol{\mu}'.\mathbf{v}^{-1}.\boldsymbol{\mu}).\lambda + (\boldsymbol{\mu}'.\mathbf{v}^{-1}.\mathbf{i}).\boldsymbol{\gamma} = \mu_T \\
& \text{similarly we premultiply with } \mathbf{i}' \\
\mathbf{i}'.\mathbf{w} &= \mathbf{i}'.\mathbf{v}^{-1}.(\lambda.\boldsymbol{\mu}{+}\gamma.\mathbf{i}) = 1 \\
\mathbf{i}'.\mathbf{w} &= (\mathbf{i}'.\mathbf{v}^{-1}.\boldsymbol{\mu}).\lambda + (\mathbf{i}'.\mathbf{v}^{-1}.\mathbf{i}).\boldsymbol{\gamma} = 1
\end{aligned}$$

Then we define

$$\begin{aligned}
A &: = \mathbf{i}'.\mathbf{v}^{-1}.\boldsymbol{\mu} \\
B &: = \boldsymbol{\mu}'.\mathbf{v}^{-1}.\boldsymbol{\mu} \\
C &: = \mathbf{i}'.\mathbf{v}^{-1}.\mathbf{i} \\
D &: = B.C - A^2
\end{aligned}$$

Note that these are all scalars and we obtain

$$\begin{aligned}
\lambda.B + \gamma.A &= \mu_T \\
\lambda.A + \gamma.C &= 1
\end{aligned}$$

We express the lagrange multiplier as

$$\begin{aligned}
\lambda &= \frac{C.\mu_T - A}{D} \\
\gamma &= \frac{B - A.\mu_T}{D}
\end{aligned}$$

We plug them into the first n optimality constraints, then $\mathbf{w} = \mathbf{v}^{-1}.(\lambda.\boldsymbol{\mu}{+}\gamma.\mathbf{i})$ becomes

$$\begin{aligned}
\mathbf{w} &= \mathbf{v}^{-1}.(\boldsymbol{\mu}.\frac{C.\mu_T - A}{D} + \mathbf{i}.\frac{B - A.\mu_T}{D})_{(n\times 1)} \\
&= \frac{1}{D}.\{(C.\mu_T).\mathbf{v}^{-1}.\boldsymbol{\mu}{-}A.\mathbf{v}^{-1}.\boldsymbol{\mu}{+}B.\mathbf{v}^{-1}.\mathbf{i} - (A.\mu_T).\mathbf{v}^{-1}.\mathbf{i}\} \\
& \text{We seperate the terms that depend on target return} \\
&= \frac{1}{D}.\{\mu_T.(C.\mathbf{v}^{-1}.\boldsymbol{\mu} - A.\mathbf{v}^{-1}.\mathbf{i}) + B.\mathbf{v}^{-1}.\mathbf{i} - A.\mathbf{v}^{-1}.\boldsymbol{\mu}\} \\
&= \mathbf{h}.\mu_T + \mathbf{g} \\
\mathbf{h}_{(n\times 1)} &: = \frac{(C.\mathbf{v}^{-1}.\boldsymbol{\mu} - A.\mathbf{v}^{-1}.\mathbf{i})}{D},\ \mathbf{g}_{(n\times 1)} := \frac{B.\mathbf{v}^{-1}.\mathbf{i} - A.\mathbf{v}^{-1}.\boldsymbol{\mu}}{D}
\end{aligned}$$

Two special cases (no expected return and 100% expected return) :

$$\begin{aligned}\mu_T &= 0 \Leftrightarrow \mathbf{w} = \mathbf{g} \\ \mu_T &= 1 \Leftrightarrow \mathbf{w} = \mathbf{g} + \mathbf{h}\end{aligned}$$

Lemma (First Seperation Theorem) The vectors g and $g + h$ span the whole frontier. So, only two portfolios are need to generate the whole frontier.

$\Rightarrow$ $\mathbf{w}_q$= $\mathbf{g} + \mathbf{h}.\mu_q$, where μ_q is the target return of portfolio q. Note that $\mathbf{w}_q$ is linear combination of $\mathbf{g}$ and $\mathbf{g} + \mathbf{h}$:

$$\mathbf{w}_q = (1 - \mu_q).\mathbf{g} + \mu_q.(\mathbf{g} + \mathbf{h}) = \mathbf{g} + \mathbf{h}.\mu_q$$

Then the portfolio choice becomes minimizing

$$\sigma_p^2 = \mathbf{w}^{'}.\mathbf{v}.\mathbf{w} = (\mathbf{g} + \mathbf{h}.\mu_q)^{'}.\mathbf{v}.(\mathbf{g} + \mathbf{h}.\mu_q)$$

The graph can be drawn using

$$\frac{\sigma_p^2}{1/C} - \frac{(\mu_T - A/C)}{D/C^2} = 1$$

Recall that the general equation for hyperbole is

$$\frac{(x - x_0)^2}{a^2} - \frac{(y - y_0)}{b^2} = 1$$

Insert here Figure 5

Looking at the graph, we see that the two asymtotes are $\mu_p = \frac{A}{C} \pm \sqrt{\frac{D}{C}}\sigma_p$ and the minimum variance portfolio is at $(0, \frac{A}{C})$

Lecture 9 / Week 5

Modern Portfolio Theory (Mean -Variance Analysis)

We will introduce a more general proposition than the previous lemma.

Proposition Any portfolio frontier can be generated using two (other not necessarily g and $g + h$) frontier portfolios

Proof Let p_1 and p_2 be two frontier portfolios. $[(w_{p_1}, w_{p_2}),(\mu_{p_1}, \mu_{p_2})]$ We want to show that we can replicate any frontier portfolio $q.(w_q, \mu_q)$, formally we need to show

$$\mu_q = \alpha\mu_{p_1} + (1 - \alpha)\mu_{p_2}$$

We showed in the previous lemma that using g and $g+h$ we can replicate $w_p = g + h\mu_p$. Using this result

$$\begin{aligned} w_q &= \alpha w_{p_1} + (1-\alpha) w_{p_2} = \alpha(g + h\mu_{p_1}) + (1-\alpha)(g + h\mu_{p_2}) = \\ &= g + h(\alpha\mu_{p_1} + (1-\alpha)\mu_{p_2}) \\ &= g + h\mu_q. \quad QED. \end{aligned}$$

Case 1: **one risky and one riskless asset.**

Next we will analyze the case where **one risky and one riskless asset.** We will denote w: the portfolio weight in risky asset. But then since it is a two asset case

$$\sigma_p^2 = w^2.\sigma^2 + (1-w)^2.\sigma_{rf}^2 + 2.w_1.(1-w).\sigma_{1rf}$$

but since we have a riskfree asset, the second and third term in the summation will be 0. Hence

$$\sigma_p^2 = w^2.\sigma^2 \Rightarrow w = \frac{\sigma_p}{\sigma}$$

Then we can plug in this result in the portfolio return

$$\begin{aligned} \mu_p &= w.\mu_1 + (1-w).r^0 \\ &= w(\mu_1 - r^0) + r^0 = \frac{\sigma_p}{\sigma}(\mu_1 - r^0) + r^0 \end{aligned}$$

we can see that what we obtain a portfolio frontier which is a straight line.

Insert here Figure 1

Case 2: **two risky and one riskless asset.**

If we have two risky asset as presented in the following figure, where the line $(r^0 r^2)$ is above $(r^0.r^1)$, then the optimizing agent should invest her money in riskless asset and second asset, since the portfolio frontier is tangent to the indifference curve at a higher level.

Insert here Figure 2

Case 3: **n risky and one riskless asset.**

The above analysis can be extended to n risky asset case with a riskless one. Then the optimizing agent would invest money on riskfree asset and **tangency portfolio**. The composition of how much money should be spent on riskfree asset and tangency portfolio (consisting of risky assets) depend on the risk aversion of the agent which represented by parameter α_i in the reduced form of the preferences.$(E(X) - \alpha_i\sigma^2(X))$ Given two agents A and B, if $\alpha_B > \alpha_A$, then the agent B is more risk averse (steeper indifference curve which will intersect portfolio frontier between r^0 and tangent portfolio→ she is lending.) This can be seen in the below figure.

Insert here Figure 3

In this third case we will denote by w^0 :*portfolio weight in riskfree asset*. Then we can transform our previous optimization problem (with matrix notation) in the following way:

$$Before\text{ Budget contraint : }(\mathbf{w}^{'}.\mathbf{i}{=}1) \Rightarrow w^0 + (\mathbf{w}^{'}.\mathbf{i}){=}1 \Rightarrow w^0 = 1 - \mathbf{w}^{'}.\mathbf{i}$$

Recall that the objective function is the variance of the portfolio, that the agent tries to minimize, but riskfree asset does not have any effect on the varience, rather it changes the constraint.

$$Before : \mathbf{w}^{'}.\boldsymbol{\mu}{=}\mu_T \ \Rightarrow w^0.r^0 + \mathbf{w}^{'}.\boldsymbol{\mu}{=}\mu_T$$

Plugging in the previous result

$$\begin{aligned}
(1 - \mathbf{w}^{'}.\mathbf{i})r^0 + \mathbf{w}^{'}.\boldsymbol{\mu} &= \mu_T \\
r^0 + \mathbf{w}^{'}.(\boldsymbol{\mu}{-}r^0.\mathbf{i})_{n\times 1} &= \mu_T
\end{aligned}$$

Then the optimization problem becomes

$$\begin{aligned}
&\min_{\mathbf{w}} \frac{1}{2}\mathbf{w}^{'}.\mathbf{v}.\mathbf{w} \\
s.t \quad \mu_T \quad = &\quad r^0 + \mathbf{w}^{'}.(\boldsymbol{\mu}{-}r^0.\mathbf{i})_{n\times 1}
\end{aligned}$$

We set up the Lagrangian

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2}\mathbf{w}^{'}.\mathbf{v}.\mathbf{w}{+}\lambda.(\mu_T{-}\ r^0{-}\mathbf{w}^{'}.(\boldsymbol{\mu}{-}r^0.\mathbf{i})) \\
FOC &: \frac{\partial \mathcal{L}}{\partial \mathbf{w}} = \mathbf{v}.\mathbf{w}{-}\lambda.(\boldsymbol{\mu}{-}r^0.\mathbf{i}) = \mathbf{0}_{(n\times 1)} \\
\frac{\partial \mathcal{L}}{\partial \lambda} &= \mu_T{-}r^0{-}\mathbf{w}^{'}.(\boldsymbol{\mu}{-}r^0.\mathbf{i})
\end{aligned}$$

We follow the same strategy as before, first we will isolate the control variable and then get rid off the lagrange multiplier, and plugging in to the constraints we will obtain the portfolio weights, return and variance;

$$\begin{aligned}
\mathbf{w} &= \lambda.\mathbf{v}^{-1}.(\boldsymbol{\mu}{-}r^0.\mathbf{i}) \\
&\quad \textit{multiply by } (\boldsymbol{\mu}{-}r^0.\mathbf{i})^{'} \\
(\boldsymbol{\mu}{-}r^0.\mathbf{i})^{'}\mathbf{w} &= \lambda.(\boldsymbol{\mu}{-}r^0.\mathbf{i})^{'}\mathbf{v}^{-1}.(\boldsymbol{\mu}{-}r^0.\mathbf{i}) \\
&\quad \text{Using optimality condition} \\
\lambda.(\boldsymbol{\mu}{-}r^0.\mathbf{i})^{'}\mathbf{v}^{-1}.(\boldsymbol{\mu}{-}r^0.\mathbf{i}) &= \mu_T -\ r^0 \\
\text{we call } (\boldsymbol{\mu}{-}r^0.\mathbf{i})^{'}\mathbf{v}^{-1}.(\boldsymbol{\mu}{-}r^0.\mathbf{i}) &= H \\
\lambda.H &= \mu_T -\ r^0 \\
\lambda &= \frac{\mu_T -\ r^0}{H}
\end{aligned}$$

We plug in λ to the optimal portfolio

$$\begin{aligned}
\mathbf{w} &= \frac{\mu_T - \ r^0}{H}.\mathbf{v}^{-1}.(\boldsymbol{\mu}{-}r^0.\mathbf{i}) \\
\text{since } \sigma_p^2 &= \mathbf{w}^{'}.\mathbf{v}.\mathbf{w} \\
&= \frac{(\mu_T - \ r^0)^2}{H^2}.(\boldsymbol{\mu}{-}r^0.\mathbf{i})^{'}.\mathbf{v}^{-1}.\mathbf{v}.\mathbf{v}^{-1}(\boldsymbol{\mu}{-}r^0.\mathbf{i}) \\
\text{using } H &= (\boldsymbol{\mu}{-}r^0.\mathbf{i})^{'}\mathbf{v}^{-1}.(\boldsymbol{\mu}{-}r^0.\mathbf{i}) \\
\sigma_p^2 &= \frac{(\mu_T - \ r^0)^2}{H} \Rightarrow \mu_T = \sqrt{H}.\sigma_p + r^0 \\
where\ \mathbf{i}^{'}\mathbf{w} &= 1 \text{ is the tangency portfolio.}
\end{aligned}$$

As we can see from the last equality the portfolio frontier is a straight line again.

In equilibrium, the demand for risky assets (weights in the tangency portfolio) should be equal to the supply which is equal to the value of the offered shares. (the number of outstanding shares × price).

To sum up recall that each agent solves the following problem

$$\begin{aligned}
& \max_z \ \mu_p - \alpha_i.\sigma_p^2 \\
where\ \mu_p &= z.\mu_T + (1-z).r^0 \\
\sigma_p^2 &= z^2.\sigma_T^2
\end{aligned}$$

In equilibrium, tangency portfolio is the market portfolio (usually S&P500)

Example T(50%,50%)

A:100	60%	40%	(20%,20%)
B:100	50%	50%	(25%,25%)

One such model is CAPM :

$$\begin{aligned}
\mu_i - r^0 &= \beta_i(\mu_m - r^0) \\
\beta_i &= \frac{cov(r_i, r_m)}{var(r_m)}
\end{aligned}$$

Recall from the previous chapter, if we add the time dimension to the general model; simplest case today and tomorrow the financial problem becomes

$$\begin{aligned}
& \max_{y^0, y} \ u(y^0) + \delta E(u(y)) \\
0 &\geq p_0^{'}(x^0 - w^0) + q.z \\
0 &\geq p_s^{'}(x^s - w^s) + r_s.z) \ \ \text{s=1,2...S}
\end{aligned}$$

If the asset market clears, we can apply the reverse decomposition and express the problem i.t.o. Arrow securities.

$$\begin{aligned} &\max_{y^0,y}\; u(y^0) + \delta E(u(y)) \\ 0 \;\geq\; & y^0 - w^0 + \sum_s \pi_s.\alpha_s(y^s - w^s) \end{aligned}$$

HARA (Hyperbolic Absolute Risk Aversion)

A quite common utility function is HARA. It is the general form of the utility and with appropriate selection of the parameters we can get other types log, power, etc.

Definition $T(w) = \frac{1}{A(w)}$ **absolute risk tolarance**.

Proposition A utility is HARA type if $T^{'}(w) = a + bw$. Note that it is linear to the wealth.

If we derive the FOC of the above financial problem

$$\begin{aligned} \frac{\partial L}{\partial y^0} &: \; u^{'}(y^0) = \lambda \\ \frac{\partial L}{\partial y^s} &: \; \pi_s \delta u^{'}(y^s) = \lambda \alpha_s \quad \text{s=1,2...S} \end{aligned}$$

If we take a HARA utility function

$$\begin{aligned} u(x) &= \frac{1}{b-1}(a+bx)^{\frac{b-1}{b}} \\ u^{'}(x) &= (a+bx)^{-\frac{1}{b}} \end{aligned}$$

Applying to the financial problem

$$\begin{aligned} (a+by^0)^{-\frac{1}{b}} &= \lambda \\ \pi_s.\delta(a+by^s)^{-\frac{1}{b}} &= \lambda.\alpha_s \end{aligned}$$

$$\begin{aligned} \sum_s^S \alpha_s.y^s &= \text{"}saving\text{"} = \theta_0 + \theta_1 \bar{w} \\ \bar{w} &= w^0 + \sum_s^S \alpha_s.w^s \end{aligned}$$

Note that the coefficients θ_0, θ_1 does not depend on the initial wealth. So in case of HARA utility, saving is linear in wealth.

We followed the mean-variance framework in portfolio choice instead of the Arrow-Debreu framework, since the former allowed us to focus on the real assets in the portfolio instead of the artificial Arrow securities.

Lecture 10 / Week 6

Asset Demand

In this course we have sofar dealt with the following problems

1. **Canonical portfolio problem** with the following assumptions
   - no consumption today
   - S states at time t=1
   - one risky asset
   - one riskfree asset
2. **Savings Problem**
   - one risky asset
   - consumption today
3. **Mean-Variance Problem**
   - no consumption today
   - many risky assets
   - one risk-free asset (we managed to have closed form solutions under certain assumptions such as quadratic utility or normal returns
4. **HARA portfolio Problem**
   - S states at time t=1
   - A-D securities instead of real financial assets. But, in this framework we need the complete market hypothesis in order to be able to do reverse decomposition. (i.e.financial assets⇒ A-D securities.)

Pricing

In this lecture our focus will shift from asset demand to pricing assets. We recall from Radner Equilibrium framework that we could use representative commodity (by splitting the problem into consumption and financial problem) and/or representative agent models, in order to be able to focus only on certain aspects such as pricing. We will focus on the representative agent model so that we can only focus on the pricing of the assets. Recall that we defined the **risk neutral probability** as

$$\tilde{\alpha} = \frac{\alpha_s}{q_0} = \frac{\alpha_s}{\beta}$$

where $\tilde{\alpha}$:risk neutral probability, α_s : A security price, $q_0(\beta : \textit{book notation})$:price of riskfree asset. Then we defined risk neutral pricing

$$q_j = q_0.\tilde{E}(r_j)$$

$\tilde{E}$ is the special measure with risk neutral probabilities. We also showed that no arbitrage opportunities $\Rightarrow \alpha >> 0$ and complete markets$\Rightarrow$unique $\tilde{\alpha}$, where unique set of risk-neutral probabilities implied unique prices, together with these two assumptions we defined **Arbitrage pricing**. Note first that here we use the term return as cashflow(payoff) and not as cashflow divided by its price as in previous models. Also recall that using composition we have

$$\begin{aligned} q_0 &= \sum_s \alpha_s \\ q_j &= \alpha.r \end{aligned}$$

Now we will define another term that will be often used in pricing of assets.

Definition The following term is defined as **Stochastic discount factor(SDF),** also known as **Pricing kernel,**

$$M_s = \frac{\alpha_s}{\pi_s}$$

Again the two assumptions hold;i.e. no arbitrage opportunities $\Rightarrow M_s >> 0, \forall s$ and complete markets$\Rightarrow$unique M_s, $\forall s$.

Recall again how we priced the risk-free bond

$$q_0 = \sum_s \alpha_s = q_0 = \sum_s \alpha_s \frac{\pi_s}{\pi_s} = \sum_s \frac{\alpha_s}{\pi_s}\pi_s = \mathbf{E(M)}$$

So we have shown that

$$\begin{aligned} q_0 &= E(M) \\ q_j &= E(M.r_j) \\ E(M.R_j) &= 1 \\ R_j^s &:= \frac{r_j^s}{q_j} \end{aligned}$$

where we exploited the fact that

$$q_j = \sum_s \alpha_s r_j^s = \sum_s \alpha_s r_j^s \frac{\pi_s}{\pi_s} = \sum_s \frac{\alpha_s}{\pi_s} r_j^s \pi_s = \sum_s M_s \pi_s r_j^s = E(M.r_j)$$

Now we recall our two period maximization problem (using A-D securities)

$$\begin{aligned} & \max_{y^0,y} \; u(y^0) + \delta.E(u(y)) \\ s.t\ 0 \quad \geq & \quad y^0 - w^0 + \sum_s \alpha_s (y^s - w^s) \\ FOC \quad : & \qquad u^{'}(y^0) = \lambda \\ \delta.\pi_s.u^{'}(y^s) \quad = & \quad \lambda.\alpha_s \qquad s = 1, 2, ...S \end{aligned}$$

So we have $S+2$ constraints (S states, first one and budget constraint). By dividing the two constraints and getting rid off the λ we have

$$\delta.\pi_s.\frac{u^{'}(y^s)}{u^{'}(y^0)} = \alpha_s \qquad s = 1, 2, ...S$$

We solved the optimization problem the same way as we did before, but this time aiming to find prices and not the asset demands. Then we impose the no-trade condition since we are dealing with representative agent model, in other words the above equality should hold with initial wealth since there is no body to trade in the economy. Then

$$\delta.\pi_s.\frac{u^{'}(w^s)}{u^{'}(w^0)} = \alpha_s \qquad s = 1, 2, ...S$$

We have linked the prices to the utility maximization problem as follows

$$M_s = \delta.\frac{u^{'}(w^s)}{u^{'}(w^0)}$$

The term on the RHS explains the utility counterpart of the SDF, at the same time we linked the problem to aggregate consumption level since it is a representative agent model and thus has empirical validity. It also explains why it has the name SDF, because we see that the marginal utility is state dependent, i.o.w stochastic, and it also a discount factor since it includes the preference parameter δ, which tells us how much the agent values her today's consumption over tomorrow's consumption.

All we need to asset the prices in general is to use either the risk neutral probabilities or to use SDF. Both are based on aggregate economy using representative agent.

Now we will explain why $\tilde{\alpha}$ is called the risk neutral probability: recall again

$$\begin{aligned}
\tilde{\alpha}_s &= \frac{\alpha_s}{q_0} \\
M_s &= \frac{\alpha_s}{\pi_s} \\
\tilde{\alpha}_s.q_0 &= \alpha_s \\
M_s.\pi_s &= \alpha_s \\
\tilde{\alpha}_s.q_0 &= M_s.\pi_s \\
\frac{\tilde{\alpha}_s}{\pi_s} &= \frac{M_s}{q_0} \\
\frac{\tilde{\alpha}_s}{\pi_s} &= \frac{M_s}{E(M)} \\
\text{since } q_0 &= E(M) \\
\frac{\tilde{\alpha}_s}{\pi_s} &= \frac{\delta.\frac{u^{'}(w^s)}{u^{'}(w^0)}}{\delta.\frac{E(u^{'}(w))}{u^{'}(w^0)}} = \frac{u^{'}(w^s)}{E(u^{'}(w))} \\
\text{since } E(M) &= q_0 = \delta.\frac{E(u^{'}(w))}{u^{'}(w^0)}
\end{aligned}$$

q_0 can be interpreted as the price of time. (price of the risk-free asset.) So we have

$$\frac{\tilde{\alpha}_s}{\pi_s} = \frac{u^{'}(w^s)}{E(u^{'}(w))} \quad (*)$$

Now suppose we have **risk neutral agent:** then still u$^{'}(x) = K > 0.(u^{"}(x) = 0)$. Plugging into formula we have

$$\frac{\tilde{\alpha}_s}{\pi_s} = \frac{K}{\sum_s \pi_s K} \Leftrightarrow \tilde{\alpha}_s = \pi_s \quad \forall s.$$

This equation in case of risk neutral agent shows why it is called risk neutral probability.

In the next part we will conduct and **comparative analysis** of equilibrium price of risk-free bond in case of *risk averse agent*: We will call the risk averse agent *pessimistic,* since *given* $u^{"} < 0$, from (*) we can see that she puts too much weight for bad states if w^s is small $\tilde{\alpha}_s > \pi_s$ and little weight for good states,i.e. if w^s is large$\Rightarrow$ $\tilde{\alpha}_s < \pi_s$. Note that here small or large weight should be understood relative to the risk neutral case.

The Equilibrium Price of time

Recall the formula that we will use to do comparative statistics of the equilibrium price of time:

$$q_0 = \delta.\frac{E(u^{'}(w))}{u^{'}(w^0)}$$

Case 1 No uncertainty-no growth in wealth:

$$w_0 = w_j^1 = ... = w_j^S$$

the first equality says that there is no growth whereas the others say that no uncertainty across states. Then we have

$$\begin{aligned} q_0 &= \delta.\frac{\sum_s \pi_s u^{'}(w^0)}{u^{'}(w^0)} = \delta \\ q_0 &= \delta \end{aligned}$$

So, the price of the risk free bond is equal to the time preference parameter.

Case 2 No uncertainty + growth in wealth,i.e.

$$w_j^1 = ... = w_j^S = (1+g)w^0 \quad \text{g>0}$$

Then we have

$$\begin{aligned} q_0 &= \delta.\frac{\sum_s \pi_s u^{'}(w^0(1+g))}{u^{'}(w^0)} \\ &\Rightarrow q_0 < \delta \end{aligned}$$

This also follows from risk averse assumption.

Case 3 Uncertainty + growth in wealth, i.e. since we have in each state higher wealth, then we say $\bar{w} = w^0(1+g)$ FSD w^0. Then

$$\begin{aligned} \bar{q}_0 &= \delta.\frac{E(u^{'}(\bar{w}))}{u^{'}(w^0)} \\ & \quad by \text{ FSD} \\ &\Rightarrow \bar{q}_0 < \delta \end{aligned}$$

Comparing case 2 and 3. we can see that it is the growth in wealth that has the same effect on the price of riskless bond regardless of uncertainty. We can see this result intuitively that, since agent is risk averse, she does not like variation in wealth and would like to smooth its consumption across states, but since it is a representative agent model, she has to consume what she is endowed with, the desire to move consumption from tomorrow to today pushes down the price below δ.(which in turn increases the return of the risk free bond.)

Case 4 Uncertainty + no growth in wealth, i.e take $w^s = w^0$, w^0 SSD $\bar{w}$ where E($\bar{w}$)=w^0.(mean preserving spread.) In this last case we will look at two different cases, the first one where

$$u^{'}(x) = a + bx$$

where u$^{'}$() is linear and u$^{"}$()=b. Then

$$q_0 = \delta.\frac{\sum_s \pi_s(a + b.\bar{w}^s)}{u^{'}(w^0)} = \delta.\frac{a + b.\sum_s \pi_s \bar{w}^s}{a + b.w^0} = \delta$$

Note that in this case, adding mean preserving spread did not change the price since no prudence. (u$^{'''}(x) = 0$). In the second case we take a utility where u$^{'''}(x) > 0 \Rightarrow$ *prudence*, formally the marginal utility is convex. Then

$$\bar{q}_0 = \delta.\frac{E(u^{'}(\bar{w}))}{u^{'}(w^0)} > \delta.\frac{u^{'}(E(\bar{w}))}{u^{'}(w^0)} = \delta$$

this inquality is a result of the jensen's inequality, which holds for convex functions. A prudent agent substitutes consumption of today with consumption of tomorrow, since the agent does not like the risk, she wants to insure herself for bad states and would like to save more in order not to be harshly affected in bad states. Again, since we are in a representative agent model, prudence will push the price of riskfree bond upwards.

Lecture 11 / Week 6

Equilibrium Price of Risk

We have derived the following equation in the previous class. Now we will combine the idea of risk neutral returns with SDF (Stocastic Discount Factor).

$$E(M.R^j) = 1 \;\; \forall j$$

recall that M is the SDF($M_s = \frac{\alpha_s}{\pi_s}$) and R($R^s_j = \frac{r^s_j}{q_j}$) is the gross return of the asset j. We will exploit the the covariance formula of two random variables

$$\begin{aligned} cov(x, y) &= E(x - E(x)).E(y - E(y)) \\ &= E(x.y) - E(x).E(y) \end{aligned}$$

So we can split $E(M.R^j)$ into two terms

$$\begin{aligned}
1 &= E(M).E(R^j) + cov(M, R^j) \\
\frac{1}{E(M)} &= \frac{E(M).E(R^j)}{E(M)} + \frac{cov(M, R^j)}{E(M)} \\
E(R^j) - \frac{1}{E(M)} &= -\frac{cov(M, R^j)}{E(M)} \\
q_0 &= E(M) \Rightarrow \frac{1}{E(M)} = R^0 \\
E(R^j) - R^0 &= -R^0.cov(M, R^j)
\end{aligned}$$

Note that on the LHS we have risk premium, where R^0 is the riskfree interest rate. Substituting the SDF with the first order condition of the portfolio problem of the representative agent, we will obtain the socalled **consumption-based asset pricing model**. (CCAPM), i.e

$$\begin{aligned}
M_s &= \delta.\frac{u^{'}(w^s)}{u^{'}(w^0)} \Rightarrow E(R^j) - R^0 = -R^0.cov(\delta.\frac{u^{'}(w)}{u^{'}(w^0)}, R^j) \\
R^0 &= \frac{u^{'}(w^0)}{\delta.E(u^{'}(w))} \Rightarrow E(R^j) - R^0 = -\frac{u^{'}(w^0)}{\delta.E(u^{'}(w))}.cov(\delta.\frac{u^{'}(w)}{u^{'}(w^0)}, R^j) \\
&\Rightarrow \quad E(R^j) - R^0 = \frac{cov(-u^{'}(w), R^j)}{E(u^{'}(w))}.\forall j \qquad (\textbf{CCAPM})
\end{aligned}$$

CCAPM basically says that if the rate of return of an asset does not covary with aggregate risk (representative agent model) , then the risk premium is zero, and the expected rate of return rate of the asset equals risk free rate. Notice that in such a case even though asset's return is stochastic, no premium will be paid for this asset specific risk since it is unrelated with the aggregate risk and can be diversified away. In an efficient allocation, such risk will not be borne by anyone in the economy, therefore it has no effect on the asset's price. On the other hand, an asset whose return covaries negatively (positively) with aggregate endowment carries negative (positive) risk premium since it gives good return bad(good) times and hedges against aggregate risk. (unfavorable return.)

We did a similar analysis in case of one risky and one riskless asset, i. e

$$E(R^j) \geq R^0 \text{ iff } cov(R^j, u^{'}(w^0.r_p^*)) \leq 0$$

The above analysis generalizes to the n asset case.

Important Special Cases

Case 1 **Risk Neutral Agent**: In this case, the utility function of the agent will be linear, hence marginal utility will be constant. Then the above formula boils down to

$$\begin{aligned} & \Rightarrow & cov(-u^{'}(w), R^j) = 0 \Rightarrow E(R^j) - R^0 = 0 \\ E(R^j) & = & R^0 \\ E(\frac{r^j}{q_j}) & = & \frac{1}{q^0} \Rightarrow q_j = q^0.E(r^j) \end{aligned}$$

The above formula says that in case of a risk neutral agent, the price of an asset just equals the expected present discounted value (PV) of the cash flow it generates, using the risk free interest rate for discounting.

Case 2 No Uncertainty in the economy (No aggregate risk): This is the case when the representative agent's income is constant across states;

$$w^0 = w^1 = ... = w^s$$

In this case regardless of the presence of growth, the SDF will be constant

$$\begin{aligned} M_s & = & \delta.\frac{u^{'}(w^s)}{u^{'}(w^0)} = \delta = q^0 \\ & \Rightarrow & E(R^j) = R^0, \ \forall j \end{aligned}$$

This shows that only aggregate risk affects prices, in other words ideosyncratic risk is diversifiable if markets are complete and has no impact on asset prices.

Quadratic Utility Representative Agent and CAPM

We suppose that there is a special asset, called m, whose return is perfectly negatively correlated with the state contingent marginal utility of the representative agent, that is to say

$$\begin{aligned} R_s^m &= -au^{'}(w^s)+b, \quad \text{a>0} \\ -u^{'}(w^s) &= \frac{R_s^m - b}{a} \end{aligned}$$

If we plug in to the **CCAPM** formula we obtain

$$E(R^j) - R^0 = \frac{cov(R_s^m, R^j)}{E(u^{'}(w))}.\frac{1}{a} \quad \forall j$$

Since this holds for all assets, it must also hold for the specific asset m, hence

$$E(R^m) - R^0 = \frac{var(R^m)}{E(u^{'}(w))}.\frac{1}{a}$$

if we divide the first by the second we get a formula without marginal utility;

$$\frac{E(R^j) - R^0}{E(R^m) - R^0} = \frac{cov(R_s^m, R^j)}{var(R^m)} \quad \forall j$$

Defining

$$\begin{aligned} \beta_j &= \frac{cov(R_s^m, R^j)}{var(R^m)} \\ E(R^j) &= R^0 + \beta_j.[E(R^m) - R^0] \end{aligned}$$

This formula is the wellknown **Capital Asset Pricing Model (CAPM)**. To be able to use this formula there must be an asset whose return is perfectly negatively correlated with the marginal utility. Suppose **m (market portfolio)** is the claim on aggregate or mean endowment , so that $r_s^m = w^s$. Let the price of this asset q_m then we have

$$R_s^m = \frac{w^s}{q_m}.$$

We will make a further assumption, that the representative agent has a quadratic utility, i.e

$$\begin{aligned} u(w) &= cw - dw^2, \quad d > 0 \\ u^{'}(w) &= c - 2dw^s \Leftrightarrow w^s = \frac{c - u^{'}(w)}{2d} \end{aligned}$$

Then plugging in into market gross return

$$R_s^m = \frac{c - u^{'}(w)}{2dq_m} = \frac{c}{2dq_m} - \frac{u^{'}(w)}{2dq_m}, \quad 2dq_m > 0$$

so, we notice that in this special case the market return is perfectly negatively correlated which justifies the use of CAPM, which in fact is a special case of CCAPM. On the one hand CAPM is a very suitable model for empirical analysis, since it includes only observable variables, that's why it is so popular and widely used. One the other hand, the assumption of quadratic utility is too strong and not realistic and the special asset which is chosen as market portfolio is usually narrowly defined (**Roll's Critique**) either equity market or some other index which obviously does not represent the whole economic activity.

Asymmetric Information

Until now we did not deal with the information issue and assumed that there is symmetric information among agents, but this is again a very strong assumption and hence models with asymmetric information are developed to cope with such information problems. To illustrate the problem we will start with the following example.

Example Assume that we have the model with the following assumptions

- two agents: i, j
- state space : $S = 5$
- complete markets
- $\pi :=$ prior distribution
- 3 periods: -1:=ex-ante, 0:=interim, 1:=ex-post

We also assume that agents receive **signal** that is random variable correlated with the states of the world. Signals y(s) are such that

$$\begin{aligned} y_i(s) &= \left\{ \begin{array}{cr} 1 & \text{if } \{s_1, s_2, s_3\} \\ 2 & \text{if } \{s_4, s_5\} \end{array} \right\} \\ y_j(s) &= \left\{ \begin{array}{cr} 1 & \text{if } \{s_1, s_2\} \\ 2 & \text{if } \{s_3, s_4, s_5\} \end{array} \right\} \end{aligned}$$

Signals can be interpreted as additional information on which state might occur; in this example if agent i receives 1 as a signal she knows that either s_1,s_2 or s_3 might occur whereas if agent j receives the same signal he will infer that s_1 or s_2 might occur. In such model, if agent sees that an asset has positive price p_3 (i.e in state 3), she will be tempted to short sell the asset in an infinite amount, since she knows that such a state will not occur given her signal(s=1) . Therefore, in a world with asymmetric information, we can not guarantee anymore the existence of an A.D equilibrium, so we have to formalize the equilibrium concept in a new framework (**Rational Expectation models,** where agents collect information about others' information directly from prices).

Now we will formalize such an economy where there is asymmetric information; assume we have S states and I agents, then the economy is summarized by individual utilities and initial wealth: $(u_i, w_i)_{i=1}^{I}$. We will have the same assumption on utility function, namely $u^{'} > 0, u^{"} < 0$, then we will define the **posterior distribution**

$$v^s(y_i) = prob(\tilde{s} = s | \tilde{y}_i = y_i)$$

We will also assume that at ex-ante stage there will be no asymmetric information, so usual competitive equilibrium can be achieved. Then we claim that there will be no trade at interim stage given the equilibrium at t=-1, so in other words, *ex-ante* Pareto optimality implies *interim* Pareto optimality. (The reasoning is the following, the agents do not want to trade with others who want to trade with them after an equilibrium achieved.)

Definition A feasible allocation x is **ex-ante Pareto optimal** if there exist no other feasible allocation x$^{'}$ such that

$$\begin{aligned} Ex-ante\ P.O &: \sum_{s=1}^{S} \pi_s u_i(x_i^{s^{'}}) \geq \sum_{s=1}^{S} \pi_s u_i(x_i^s)\ \forall i \\ Interim\ P.O &: \sum_{s=1}^{S} v^s u_i(x_i^{s^{'}}) \geq \sum_{s=1}^{S} v^s u_i(x_i^s) \end{aligned}$$

with at least one strict inequality.

So, the **No-trade Theorem** says that, ex-ante Pareto optimality⇒interim Pareto optimality. We will mainly focus on the interim stage where there is asymmetric information.

We will consider the *standard consumption problem; there is a single good and no consumption at t=0. Then utility maximization becomes*

$$\begin{array}{c} \max_{x_i} \sum_{s=1}^{S} v^s(y_i) u_i(x_i^s) \\ s.t\ \ \sum_{s=1}^{S} p^s x_i^s \leq \sum_{s=1}^{S} p^s w_i^s \end{array}$$

Recall the equilibrium defined a la A.D:

1. x^* solves UMP
2. $\sum_{s=1}^{S} x_i^s \leq \sum_{s=1}^{S} w_i^s, \forall s$

The novelty in this new model is the probabilities are replaced with posterior distributions. We know that in expected utility representation prices depend on probibility π, *so* we would expect that prices would depend on signals (posterior distribution.);

$$p^* = \phi(y_1, ...y_I)$$

but this will not be the same equilibrium prices as in A.D case. Let's assume that price functional is invertible, then the agents can see the signals and revise their probabilities

$$y = (y_1, ...y_I) \in \phi^{-1}(p^*)$$

So they will update the probabilities

$$v^s(y_i, \phi^{-1}) = prob(s = s|y_i \ and \ y_{-i})$$

where $y_{-i} := signals \ of \ others.$

Lecture 12 / Week 6

Green Lucas Equilibrium

Definition A **Green-Lucas equilibrium** is a price functional :$\phi^*()$: $\mathbb{R}^I \to \mathbb{R}$ such that $\sum_{i=1}^{I} x_i^s(y_i, p^* = \phi^*(y)) = \sum_{i=1}^{I} w_i^s \ \ \forall s,$ (demand=aggregate endowment), where x_i maximizes expected utility.

We will make use of the following two mathematical results to construct our new equilibrium concept.

1. Lognormal Distribution: Let X$\sim$ $N(\mu, \sigma^2)$. Then Y=exp(X) is lognormally distributed.

   $$E(Y) = E(\exp(X)) > \exp(E(X))$$

   recall the trick that E(u(X))<u(E(X)) if u is concave, so the Jensen's Inequality is the opposite case. Then

   $$E(Y) = E(\exp(X)) = \exp(\mu + \frac{1}{2}\sigma^2)$$

2. Projection Theorem: Let (X,Y) be jointly binormal, i.e.

$$\begin{bmatrix} X \\ Y \end{bmatrix} \sim N\left(\begin{pmatrix} \mu_X \\ \mu_Y \end{pmatrix}, \begin{bmatrix} \sigma_X^2 & \sigma_{XY} \\ \sigma_{XY} & \sigma_Y^2 \end{bmatrix}\right)$$

Then the conditional

$$\begin{aligned} X|Y &\sim N(\mu_{X|Y}, \sigma^2_{X|Y}) \\ \mu_{X|Y} &= \mu_X + \frac{\sigma_{XY}}{\sigma_Y^2}(Y - \mu_Y) \\ \sigma^2_{X|Y} &= \sigma_X^2 - \frac{(\sigma_{XY})^2}{\sigma_y^2} \end{aligned}$$

Grossman Economy

We have the following assumptions:

- two dates t=0,1
- no consumption today
- two assets

$$\begin{aligned} \text{riskfree asset r}_0 &= 1 \\ \text{risky asset} &: \widetilde{f} \sim N(\mu_f, \sigma_f^2) \end{aligned}$$

  where f :fundamental value.

- agents: continuum of investors [0,1]:

$$\begin{aligned} \text{CARA utility} &: u_i(w_i) = -\exp(-\gamma_i w_i)\ \forall i \\ \text{endowments} &: \overline{x}_i \quad \forall i \\ \overline{x} &: = \int_0^1 \overline{x}_i\ di. \end{aligned}$$

- asymmetric information

$$\begin{aligned} \lambda &\in (0,1) : \text{informed agents: } \mathrm{y}_{i,I}{=}\widetilde{f} + \varepsilon_i,\ \varepsilon_i \sim \mathrm{N}(0,\sigma^2_{\varepsilon_i}), \varepsilon_i \perp \varepsilon_j \\ (1-\lambda) &\in (0,1) : \text{ uninformed agents: no signal} \Rightarrow y_{i,U} = \{\emptyset\} \end{aligned}$$

- we will further make no heterogeniety assumption w.l.o.g, then

$$\begin{aligned} \gamma_i &= \gamma \quad \forall i \\ y_{i,I} &= y_I \ \ \forall i \text{ (same information for informed)} \\ \sigma^2_{\varepsilon_i} &= \sigma^2_{\varepsilon} \text{ (same signal)} \end{aligned}$$

Utility Maximization Problem

$$\begin{aligned} \max_{x_{i,0},x_i} \ E(u(w_i)) &= E(-\exp(-\gamma w_i)\ ) \\ s.t\ \ px_i + q_0 x_{i,0} &= p.\bar{x}_i \Leftrightarrow x_{i,0} = \frac{p.(\bar{x}_i - x_i)}{q_0} \\ w_i &= \widetilde{f}\ x_i + r_0 x_{i,0} \\ w_i &= \widetilde{f}\ x_i + \frac{r_0}{q_0} p.(\bar{x}_i - x_i) \end{aligned}$$

this is the portfolio problem without the asymmetric information. We set $q_0 = 1$, i.e. we normalize the prices since only the relative prices matter. We will further assume that $\bar{x}_i = 0 \ \ \forall i$. (mathematical simplification).

In the following we willl describe the maximization problem for both types of agents, so we introduce asymmetric information, below are the UMP for uninformed and informed agent respectively

Uninformed agent

$$\begin{aligned} & \max_{x_u} \ \ E(-\exp(-\gamma w_u)\ ) \\ s.t\ w_u &= (\widetilde{f} - p)x_u \end{aligned}$$

Informed agent

$$\begin{aligned} & \max_{x_I} \ \ E(-\exp(-\gamma w_I)\ |y_I) \\ s.t\ w_I &= (\widetilde{f} - p)x_I \end{aligned}$$

First we solve the problem for uninformed agent given our assumptions

$$\begin{aligned} w_u &\sim N(x_u(u_f - p), x_u^2\sigma_f^2) \\ -\gamma w_u &\sim N(-\gamma x_u(u_f - p), \gamma^2 x_u^2\sigma_f^2) \end{aligned}$$

Then the problem becomes

$$\begin{aligned} \max_{x_u} \ \ E(-\exp(-\gamma w_u)\ ) &= \min_{x_u} \ E(\exp(-\gamma w_u)\ ) = \min_{x_u} \ \exp(-\gamma x_u(u_f - p) + \frac{1}{2}\ \gamma^2 x_u^2\sigma_f^2) \\ &=^{mon.\ of\ \exp} \max_{x_u}(x_u(u_f - p) - \frac{1}{2}\ \gamma x_u^2\sigma_f^2) \end{aligned}$$

so CARA assumption with lognormal distribution result provides tractabilility, i.e. closed form sollution to the problem. Now we solve the simplified problem

$$\begin{aligned} & \max_{x_u}(x_u(u_f - p) - \frac{1}{2}\ \gamma x_u^2\sigma_f^2) \\ FOC &: \ \mu_f - p - \frac{2}{2}\gamma x_u\sigma_f^2 = 0 \Rightarrow x_u^* = \frac{\mu_f - p}{\gamma\sigma_f^2} \end{aligned}$$

this is the closed form solution for the uninformed. With some little manipulation we can show that it is a similar result we have found earlier in class:

$$\begin{aligned} x_u &= \frac{p.(\frac{u_f}{p} - 1)}{p^2\gamma\frac{\sigma_f^2}{p^2}} \Rightarrow p.x_u = \frac{(\frac{u_f}{p} - 1)}{\gamma\frac{\sigma_f^2}{p^2}} = \frac{E(\frac{\tilde{f}}{p}) - 1}{\gamma var(\frac{\tilde{f}}{p})} = \\ &= \frac{E(r) - 1}{\gamma var(r)}, \quad \frac{\tilde{f}}{p} : \text{gross return on stock} \\ &= \frac{E(r) - r^0}{\gamma var(r - r^0)} \end{aligned}$$

Note that what we have calculated is exactly the same as what we have found as approximation for small risk given CARA assumption,i.e.

$$\begin{aligned} z^* &= \frac{E(r-r^0)}{var(r-r^0).A(w_0.r^0)} \\ CARA &\Rightarrow A(w_0.r^0)=\gamma. \end{aligned}$$

For the informed agent we recall our assumption and exploit the projection theorem

$$\begin{aligned} & w_I|y_I \\ w_I|y_I &\sim N(x_I(u_{f|y_I}-p), x_I^2\sigma^2_{f|y_I}) \\ \widetilde{f} &\sim N(\mu_f, \sigma_f^2) \\ \mathrm{y}_{i,I} &= \widetilde{f} + \varepsilon_i,\ \varepsilon_i \sim \mathrm{N}(0,\sigma^2_{\varepsilon_i}), \varepsilon_i \perp \varepsilon \\ \begin{bmatrix} f \\ y_I \end{bmatrix} &\sim N\left(\begin{pmatrix} \mu_f \\ \mu_f \end{pmatrix}, \begin{bmatrix} \sigma_f^2 & \sigma_f^2 \\ \sigma_f^2 & \sigma_f^2+\sigma_\varepsilon^2 \end{bmatrix}\right) \\ \widetilde{f}|y_I &\sim N(\mu_{f|y_I}, \sigma^2_{f|y_I}) \\ \mu_{f|y_I} &= \mu_f + \frac{\sigma_f^2}{\sigma_f^2+\sigma_\varepsilon^2}(y_I-\mu_f) \\ \sigma^2_{f|y_I} &= \sigma_f^2 - \sigma_f^2.(\sigma_f^2+\sigma_\varepsilon^2)^{-1} \end{aligned}$$

Under these assumptions we obtain the an analogous solution as before

$$x_I^* = \frac{\mu_{f|y_I}-p}{\gamma\sigma^2_{f|y_I}}$$

Then we impose the market clearing condition

$$MCC: \int_0^\lambda x_I\ di + \int_\lambda^1 x_u di = 0$$

The RHS comes from the assumption that there is no initial endowment in risky asset. LHS is the aggregate demand.

$$\begin{aligned} u_i &= u_j \\ \lambda\frac{\mu_{f|y_I}-p}{\sigma^2_{f|y_I}} + \frac{(\mu_f-p)(1-\lambda)}{\sigma_f^2} &= 0 \\ \lambda.\sigma_f^2(\mu_{f|y_I}-p) + (1-\lambda)\sigma^2_{f|y_I}(\mu_f-p) &= 0 \\ p(\lambda\sigma_f^2 + (1-\lambda)\sigma^2_{f|y_I}) &= \lambda\sigma_f^2\mu_{f|y_I} + (1-\lambda)\sigma^2_{f|y_I}\mu_f \\ p^* &= \frac{\lambda\sigma_f^2\mu_{f|y_I} + (1-\lambda)\sigma^2_{f|y_I}\mu_f}{(\lambda\sigma_f^2 + (1-\lambda)\sigma^2_{f|y_I})} = p(y_I) \end{aligned}$$

what we found is **FRREE**. (fully revealing rational expectations equilibrium.) One can analyse the limiting cases as $\lambda \to 1$ or $\lambda \to 0$. We can see that the A.D equilibrium concept does not serve our purpose anymore, since uninformed agents can see the price and infer the signal of the informed ones, this in turn will change the demand of the uninformed and an equilibrium will not be possible. (Grossman-Stiglitz paradox: what information can prices carry, if noone pays for the information.)

Definition **Rational expectation equilibrium** (REE) is a set of trades $\{\mathrm{x}_I(y_I, p), \mathrm{x}_u(p)\}$ and the price functional $\mathrm{p}(\mathrm{y}_I)$ s.t

1. $\mathrm{x}_I(y_I, p)$ solves

$$\max_{x_I} \; E(-\exp(-\gamma \; (\widetilde{f} - p)x_I)|y_I, p)$$

   $\mathrm{x}_u(p)$ solves

$$\max_{x_u} \; E(-\exp(-\gamma \; (\widetilde{f} - p)x_u)|p)$$

2. M.C.C:

$$\int_0^\lambda x_I(y_I, p) \; di + \int_\lambda^1 x_u(p) \; di = 0$$

One follows the following strategy to solve the REE: First conjecture a price functional and start with the competitive equilibrium, solve the UMP for both types of agents that solves MCC.

Example We conjecture the price functional

$$\begin{array}{rcl}
p(y_I) & = & a + by_I \\
\text{Uninformed} & : & \max\limits_{x_u} \; E(-\exp(-\gamma \; w_u)|y_I) \\
\text{Informed} & : & \max\limits_{x_I} \; E(-\exp(-\gamma \; w_I)|y_I) \\
x_u^* & = & \dfrac{\mu_{f|y_I} - p}{\gamma\sigma^2_{f|y_I}} = x_I^* \\
MCC & : & \dfrac{\mu_{f|y_I} - p}{\gamma\sigma^2_{f|y_I}} = 0 \\
p^{**} & = & \mu_{f|y_I} = \mu_f + \dfrac{\sigma_f^2}{\sigma_f^2 + \sigma_\varepsilon^2}(y_f - \mu_f) \\
b & = & \dfrac{\sigma_f^2}{\sigma_f^2 + \sigma_\varepsilon^2} \\
a & = & \mu_f(1 - b)
\end{array}$$