Lecture 1 / Week 1

Basics

We shortly review the basic consumer theory on preferences, since most of the models in asset pricing use the main assumptions of this theory. There is the set $\mathcal{X}$ which gives us what the consumer can choose and there is preference relation defined on that set. $(\mathcal{X}, \succeq)$ Most of the time in this class our set will look like $\mathcal{X} \equiv \mathbb{R}_+^L$.

Definition 1 *The* ***preference relation*** $\succeq$ *is defined as follows:*

$$\succeq \text{ preference relation: (at least as good as):}$$
$$\succ: x, y \in \mathcal{X} : x \succ y : x \succeq y \text{ but not } y \succeq x \text{ (strictly better)}$$
$$\sim: x, y \in \mathcal{X} : \ x \sim y : \ x \succeq y \text{ and } y \succeq x \text{ (indifference)}$$

We need to impose certain assumptions on preference relationship that we will assume to be true in this course unless stated otherwise:

1. $\succeq$ is **rational**: The rationality of preference relation is defined over completeness and transitivity:

   $$\begin{aligned} \text{completeness} &: \quad x, y \in \mathcal{X} : x \succeq y \text{ or } y \succeq x \text{, or both.} \\ & \quad \text{(we should be able to compare all elements of our set)} \\ \text{transitivity} &: \quad x, y, z \in \mathcal{X} : x \succeq y \ \text{ and } y \succeq z \Rightarrow \ x \succeq z. \\ & \quad \text{(we should be consistent in our ranking)} \end{aligned}$$

2. $\succeq$ is **continuous**

   These two conditions together guarantee that we have a utility function s.t $\Rightarrow x \succeq y \Leftrightarrow u(x) \geq u(y)$: where $u(\mathbb{R}_+^L) \to \mathbb{R}$.

3. $\succeq$ is **monotone (strictly)**: This condition tells us the first derivative is increasing. Intuitively,it imposes that goods are desirable. (more is better!)

4. $\succeq$ is **convex (strictly):** We have concave indifference curves. Intuitively, it's good to mix the bundles, so agents appreciate diversity.

We also impose the additional mathematical condition on utility function that u(x) is twice continuously differentiable ($\mathcal{C}^2$). Under these conditions we have well behave agents (concave utility functions) and finally we can deal with utility maximization.

$$\max_{x\geq 0}\ u(x)$$
$$s.t\ \ p^T(x-\omega)\leq 0 \text{ or}$$
$$p^Tx\leq W=p^T\omega$$
ω is initial endowment and W is inital wealth.

Note that once we have the desirability condition, the budget constraint will be binding, so it will be an equality constraint. To solve this problem, we can set up the Lagrangian function. (In inequality case, we have to use Kuhn-Tucker conditions)

$$\mathcal{L}(x,\lambda)=u(x)+\lambda(W-p^Tx)$$
$$FOC: \quad \frac{\partial u(x)}{\partial x_l}-\lambda p_l=0,\ \forall\ l=1,2,..L$$
$$W-p^Tx=0$$

The above system caracterizes the solution: in vector form: $\nabla u(x^*)=\lambda$ p (Note that p is a vector with l components). So the gradient of the utility function at optimum, should be collinear with the prices vector. One can also see it graphically. (Orthogonality condition to budget constraint)

Insert here Figure 1

Economy

Definition 2 *The **economy** will be caracterized as a collection of choice sets, preferences and initial endowments over the agents.*

$$\{\mathcal{X}_i,\succeq_i,\omega_i\}_i^I=\{\mathcal{X}_i,u()_i,\omega_i\}_i^I$$

Definition 3 *x={$x_1,x_2,....,x_I$} is called an allocation. An allocation x is **feasible** iff:*

$$\Sigma_{i=1}^I x_i\leq\Sigma_{i=1}^I\omega_i$$

In words, an allocation is feasible if and only if the aggregate consumption in the economy is not more than aggregate endowment. (Note: w.l.o.g, we have a pure exchange economy, so no production function.) Again, if we have the monotonicity (desirability condition) this inequality should hold with equality. ($\Sigma_{i=1}^I x_i=\Omega=\Sigma_{i=1}^I\omega_i$)

Definition 4 *A **Walrasian equilibrium** is caracterized by the optimal allocation and the optimum price vector: (x^* , p^*). It is attained if the following two conditions are satisfied:*

1. (x^* , p^*) solves the UMP. (Utility Maximization Problem)
2. x^* is feasible. (As defined above)

So the Walrasian equilibruim says that in such an economy, all the agents should maximize their utility (Condition 1) and the markets should clear.(Condition 2) There is a quiete extensive literature on existence, uniqueness and properties of this equilibrium in general equilibrium literature, but in this course we will not focus on them.

Proposition Let $\{\mathcal{X}_i, u()_i, \omega_i\}_i^I$ be a "standard economy" and suppose that $\Sigma_{i=1}^I \omega_i \gg 0$. (positive endowment requirement). Then there exists a Walrasian equilibrium (x^*, p^*). We will not prove this proposition, but just use the results.

Definition 5 *A feasible allocation x is* ***Pareto optimal (efficient)*** *if there is no other allocation $x^{'}$ (feasible) such that $u(x_i^{'}) \geq u(x_i)$ $\forall$ i and $u(x_i^{'}) > u(x_i)$ for at least one agent. So in other words, there is no alternative way to allocate resources that makes some agents better off wihout making some other agent worse off. (Recall the example of voting with unanimaty).*

FIRST WELFARE THEOREM

FWT: Let (x^*, p^*) be a Walrasian equilibrium for the economy $\{\mathcal{X}_i, u()_i, \omega_i\}_i^I$. Then the allocation x^* is Pareto efficient.

This theorem helps us to exploit the representative agent models. In such a set-up, the prices of single agent economy turn out to be the same as the prices in Walrasian equilibrium. We lose some important information on the distribution of consumption among agents, but in return we have the simplification through same prices.

Social Welfare Function: : $\mathrm{W}(u_1(x_1), u_2(x_2), ...u_I(x_I)) = \frac{1}{I}\Sigma_{i=1}^I \sigma_i u_i(x_i)$, where $(\sigma_1, \sigma_2, ..., \sigma_I) > 0$. This social welfare function depends linearly(positive weights, σ_i) on the individual utilities of agents, where it is normalized by the number of agents. (per capita.) A Pareto optimum allocation can be implemented with a suitable choice of weights. The question is which choice of weights allow us to replicate the Walrasian equilibrium. The following two propositions give us the answer to this question.

Proposition 6 *Let x be the Pareto efficient allocation. Then x can be implemented through*

$$\max_y \frac{1}{I}\Sigma_{i=1}^{I}\sigma_i u_i(x_i)$$
$$s.t \quad \Sigma_{i=1}^{I}(z - y_i) \geq 0 \text{ (feasibility constraint)}$$
$$where \ z = \frac{\Omega}{I} \text{ (mean endowment)}$$

This is the so called **Social Planner Problem (SPP).** The social planner wants to maximize the social welfare function given the feasibility constraint, which tells us that the aggregate consumption cannot exceed aggregate endowment. Imposing the desirability condition will result in an equality constraint.

Recall the Walrasian equilibrium:

$$\nabla u(x_i^*) = \lambda_i \ p \quad \forall \ i = 1, 2...., I$$
$$p^* = \frac{\nabla u(x_1^*)}{\lambda_1} = \frac{\nabla u(x_2^*)}{\lambda_2} = ... = \frac{\nabla u(x_I^*)}{\lambda_I}$$

Note that all gradiants are constant across agents. This observation leads us to the following proposition.

Proposition 7 *Let* $(x^*\ ,\ p^*)$ *be a Walrasian equilibrium and* $\{\lambda_i\}_{i=1}^{I}$ *be the Lagrange multiplier of the UMP. Then* x^* *can be implemented with* $\sigma_i = \frac{1}{\lambda_i}\ \forall\ i$ *in SPP.*