Lecture 10 / Week 6

Asset Demand

In this course we have sofar dealt with the following problems

1. **Canonical portfolio problem** with the following assumptions
   - no consumption today
   - S states at time t=1
   - one risky asset
   - one riskfree asset
2. **Savings Problem**
   - one risky asset
   - consumption today
3. **Mean-Variance Problem**
   - no consumption today
   - many risky assets
   - one risk-free asset (we managed to have closed form solutions under certain assumptions such as quadratic utility or normal returns
4. **HARA portfolio Problem**
   - S states at time t=1
   - A-D securities instead of real financial assets. But, in this framework we need the complete market hypothesis in order to be able to do reverse decomposition. (i.e.financial assets$\Rightarrow$ A-D securities.)

Pricing

In this lecture our focus will shift from asset demand to pricing assets. We recall from Radner Equilibrium framework that we could use representative commodity (by splitting the problem into consumption and financial problem) and/or representative agent models, in order to be able to focus only on certain aspects such as pricing. We will focus on the representative agent model so that we can only focus on the pricing of the assets. Recall that we defined the **risk neutral probability** as

$$\tilde{\alpha} = \frac{\alpha_s}{q_0} = \frac{\alpha_s}{\beta}$$

where $\tilde{\alpha}$:risk neutral probability, α_s : A security price, $q_0(\beta : book\ notation)$:price of riskfree asset. Then we defined risk neutral pricing

$$q_j = q_0.\tilde{E}(r_j)$$

$\tilde{E}$ is the special measure with risk neutral probabilities. We also showed that no arbitrage opportunities $\Rightarrow \alpha >> 0$ and complete markets$\Rightarrow$unique $\tilde{\alpha}$, where unique set of risk-neutral probabilities implied unique prices, together with these two assumptions we defined **Arbitrage pricing**. Note first that here we use the term return as cashflow(payoff) and not as cashflow divided by its price as in previous models. Also recall that using composition we have

$$\begin{aligned} q_0 &= \sum_s \alpha_s \\ q_j &= \alpha.r \end{aligned}$$

Now we will define another term that will be often used in pricing of assets.

Definition The following term is defined as **Stochastic discount factor(SDF),** also known as **Pricing kernel,**

$$M_s = \frac{\alpha_s}{\pi_s}$$

Again the two assumptions hold;i.e. no arbitrage opportunities $\Rightarrow M_s >> 0, \forall s$ and complete markets$\Rightarrow$unique M_s, $\forall s$.

Recall again how we priced the risk-free bond

$$q_0 = \sum_s \alpha_s = q_0 = \sum_s \alpha_s \frac{\pi_s}{\pi_s} = \sum_s \frac{\alpha_s}{\pi_s} \pi_s = \mathbf{E}(\mathbf{M})$$

So we have shown that

$$\begin{aligned} q_0 &= E(M) \\ q_j &= E(M.r_j) \\ E(M.R_j) &= 1 \\ R_j^s &= := \frac{r_j^s}{q_j} \end{aligned}$$

where we exploited the fact that

$$q_j = \sum_s \alpha_s r_j^s = \sum_s \alpha_s r_j^s \frac{\pi_s}{\pi_s} = \sum_s \frac{\alpha_s}{\pi_s} r_j^s \pi_s = \sum_s M_s \pi_s r_j^s = E(M.r_j)$$

Now we recall our two period maximization problem (using A-D securities)

$$\max_{y^0, y} \ u(y^0) + \delta.E(u(y))$$

$$\begin{aligned} s.t\ 0 &\geq y^0 - w^0 + \sum_s \alpha_s(y^s - w^s) \\ FOC &: \quad u^{'}(y^0) = \lambda \\ \delta.\pi_s.u^{'}(y^s) &= \lambda.\alpha_s \quad s = 1, 2, ...S \end{aligned}$$

So we have $S+2$ constraints (S states, first one and budget constraint). By dividing the two constraints and getting rid off the λ we have

$$\delta.\pi_s.\frac{u^{'}(y^s)}{u^{'}(y^0)} = \alpha_s \qquad s = 1, 2, ...S$$

We solved the optimization problem the same way as we did before, but this time aiming to find prices and not the asset demands. Then we impose the no-trade condition since we are dealing with representative agent model, in other words the above equality should hold with initial wealth since there is no body to trade in the economy. Then

$$\delta.\pi_s.\frac{u^{'}(w^s)}{u^{'}(w^0)} = \alpha_s \qquad s = 1, 2, ...S$$

We have linked the prices to the utility maximization problem as follows

$$M_s = \delta.\frac{u^{'}(w^s)}{u^{'}(w^0)}$$

The term on the LHS explains the utility counterpart of the SDF, at the same time we linked the problem to aggregate consumption level since it is a representative agent model and thus has empirical validity. It also explains why it has the name SDF, because we see that the marginal utility is state dependent, i.o.w stochastic, and it also a discount factor since it includes the preference parameter δ, which tells us how much the agent values her today's consumption over tomorrow's consumption.

All we need to asset the prices in general is to use either the risk neutral probabilities or to use SDF. Both are based on aggregate economy using representative agent.

Now we will explain why $\tilde{\alpha}$ is called the risk neutral probability: recall again

$$\begin{aligned}
\tilde{\alpha}_s &= \frac{\alpha_s}{q_0} \\
M_s &= \frac{\alpha_s}{\pi_s} \\
\tilde{\alpha}_s.q_0 &= \alpha_s \\
M_s.\pi_s &= \alpha_s \\
\tilde{\alpha}_s.q_0 &= M_s.\pi_s \\
\frac{\tilde{\alpha}_s}{\pi_s} &= \frac{M_s}{q_0} \\
\frac{\tilde{\alpha}_s}{\pi_s} &= \frac{M_s}{E(M)} \\
\text{since } q_0 &= E(M) \\
\frac{\tilde{\alpha}_s}{\pi_s} &= \frac{\delta.\frac{u^{'}(w^s)}{u^{'}(w^0)}}{\delta.\frac{E(u^{'}(w))}{u^{'}(w^0)}} = \frac{u^{'}(w^s)}{E(u^{'}(w))} \\
\text{since } E(M) &= q_0 = \delta.\frac{E(u^{'}(w))}{u^{'}(w^0)}
\end{aligned}$$

q_0 can be interpreted as the price of time. (price of the risk-free asset.) So we have

$$\frac{\tilde{\alpha}_s}{\pi_s} = \frac{u^{'}(w^s)}{E(u^{'}(w))} \quad (*)$$

Now suppose we have **risk neutral agent:** then still $\mathrm{u}^{'}(x) = K > 0.(u^{"}(x) = 0)$. Plugging into formula we have

$$\frac{\tilde{\alpha}_s}{\pi_s} = \frac{K}{\sum_s \pi_s K} \Leftrightarrow \tilde{\alpha}_s = \pi_s \quad \forall s.$$

This equation in case of risk neutral agent shows why it is called risk neutral probability.

In the next part we will conduct and **comparative analysis** of equilibrium price of risk-free bond in case of *risk averse agent*: We will call the risk averse agent *pessimistic,* since *given* $u^{"} < 0$, from (*) we can see that she puts too much weight for bad states if w^s is small $\tilde{\alpha}_s > \pi_s$ and little weight for good states,i.e. if w^s is large$\Rightarrow \tilde{\alpha}_s < \pi_s$. Note that here small or large weight should be understood relative to the risk neutral case.

The Equilibrium Price of time

Recall the formula that we will use to do comparative statistics of the equilibrium price of time:

$$q_0 = \delta.\frac{E(u^{'}(w))}{u^{'}(w^0)}$$

Case 1 No uncertainty-no growth in wealth:

$$w_0 = w_j^1 = ... = w_j^S$$

the first equality says that there is no growth whereas the others say that no uncertainty across states. Then we have

$$\begin{aligned} q_0 &= \delta.\frac{\sum_s \pi_s u^{'}(w^0)}{u^{'}(w^0)} = \delta \\ q_0 &= \delta \end{aligned}$$

So, the price of the risk free bond is equal to the time preference parameter.

Case 2 No uncertainty + growth in wealth,i.e.

$$w_j^1 = ... = w_j^S = (1+g)w^0 \quad \text{g>0}$$

Then we have

$$\begin{aligned} q_0 &= \delta.\frac{\sum_s \pi_s u^{'}(w^0(1+g))}{u^{'}(w^0)} \\ &\Rightarrow q_0 < \delta \end{aligned}$$

This also follows from risk averse assumption.

Case 3 Uncertainty + growth in wealth,i.e. since we have in each state higher wealth than we say $\bar{w} = w^0(1+g)$ FSD w^0. Then

$$\begin{aligned} \bar{q}_0 &= \delta.\frac{E(u^{'}(\bar{w}))}{u^{'}(w^0)} \\ & \quad by \text{ FSD} \\ &\Rightarrow \bar{q}_0 < \delta \end{aligned}$$

Comparing case 2 and 3. we can see that it is the growth in wealth that has the same effect on the price of riskless bond regardless of uncertainty. We can see this result intuitively that, since agent is risk averse, she does not like variation in wealth and would like to smooth its consumption across states, but since it is a representative agent model, she has to consume what she is endowed with, the desire to move consumption from tomorrow to today pushes down the price below δ.(which in turn increases the return of the risk free bond.)

Case 4 Uncertainty + no growth in wealth,i.e take $w^s = w^0$, w^0 SSD $\bar{w}$ where E($\bar{w}$)=w^0.(mean preserving spread.) In this last case we will loo at two different cases, the first one where

$$u^{'}(x) = a + bx$$

where u$^{'}$() is linear and u"()=b. Then

$$q_0 = \delta.\frac{\sum_s \pi_s(a + b.\bar{w}^s)}{u^{'}(w^0)} = \delta.\frac{a + b.\sum_s \pi_s \bar{w}^s}{a + b.w^0} = \delta$$

Note that in this case, adding mean preserving spread did not change the price since no prudence. (u$^{'''}(x) = 0$). In the second case we take a utility where u$^{'''}(x) > 0 \Rightarrow$ *prudence*, formally the marginal utility is convex. Then

$$\bar{q}_0 = \delta.\frac{E(u^{'}(\bar{w}))}{u^{'}(w^0)} > \delta.\frac{u^{'}(E(\bar{w}))}{u^{'}(w^0)} = \delta$$

this inquality is a result of the jensen's inequality, which holds for convex functions. A prudent agent substitutes consumption of today with consumption of tomorrow, since the agent does not like the risk, she wants to insure herself for bad states and would like to save more in order not to be harshly affected in bad states. Again, since we are in a representative agent model, prudence will push the price of riskfree bond upwards.