Lecture 11 / Week 6

Equilibrium Price of Risk

We have derived the following equation in the previous class. Now we will combine the idea of risk neutral returns with SDF (Stocastic Discount Factor).

$$E(M.R^j) = 1 \quad \forall j$$

recall that M is the SDF($M_s = \frac{\alpha_s}{\pi_s}$) and R($R_j^s = \frac{r_j^s}{q_j}$) is the gross return of the asset j. We will exploit the the covariance formula of two random variables

$$\begin{aligned} cov(x,y) &= E(x - E(x)).E(y - E(y)) \\ &= E(x.y) - E(x).E(y) \end{aligned}$$

So we can split $E(M.R^j)$ into two terms

$$\begin{aligned} 1 &= E(M).E(R^j) + cov(M, R^j) \\ \frac{1}{E(M)} &= \frac{E(M).E(R^j)}{E(M)} + \frac{cov(M, R^j)}{E(M)} \\ E(R^j) - \frac{1}{E(M)} &= -\frac{cov(M, R^j)}{E(M)} \\ q_0 &= E(M) \Rightarrow \frac{1}{E(M)} = R^0 \\ E(R^j) - R^0 &= -R^0.cov(M, R^j) \end{aligned}$$

Note that on the LHS we have risk premium, where R^0 is the riskfree interest rate. Substituting the SDF with the first order condition of the portfolio problem of the representative agent, we will obtain the socalled **consumption-based asset pricing model**. (CCAPM), i.e

$$\begin{aligned} M_s &= \delta.\frac{u'(w^s)}{u'(w^0)} \Rightarrow E(R^j) - R^0 = -R^0.cov(\delta.\frac{u'(w)}{u'(w^0)}, R^j) \\ R^0 &= \frac{u'(w^0)}{\delta.E(u'(w))} \Rightarrow E(R^j) - R^0 = -\frac{u'(w^0)}{\delta.E(u'(w))}.cov(\delta.\frac{u'(w)}{u'(w^0)}, R^j) \\ &\Rightarrow E(R^j) - R^0 = \frac{cov(-u'(w), R^j)}{E(u'(w))}.\forall j \quad (\textbf{CCAPM}) \end{aligned}$$

CCAPM basically says that if the rate of return of an asset is not correlated with aggregate risk (representative agent model) , then the risk premium is zero, and the expected rate of return rate of the asset equals risk free rate. Notice that in such a case even though asset's return is stochastic, no premium will be paid for this asset specific risk since it is unrelated with the aggregate risk and can be diversified away. In an efficient allocation, such risk will not be borne by anyone in the economy, therefor e it has no effect on the asset's price. On the other hand, an asset whose return covaries negatively (positively) with aggregate endowment carries negative (positive) risk premium since it gives good return bad(good) times and hedges against aggregate risk. (unfavorable return.)

We did a similar analysis in case of one risky and one riskless asset, i. e

$$E(R^j) \geq R^0 \text{ iff } cov(R^j, u^{'}(w^0.r_p^*)) \leq 0$$

The above analysis generalizes to the n asset case.

Important Special Cases

Case 1 Risk Neutral Agent: In this case, the utility function of the agent will be linear, hence marginal utility will be constant. Then the above formula boils down to

$$\begin{aligned} &\Rightarrow \quad cov(-u^{'}(w), R^j) = 0 \Rightarrow E(R^j) - R^0 = 0 \\ E(R^j) &= R^0 \\ E(\frac{r^j}{q_j}) &= \frac{1}{q^0} \Rightarrow q_j = q^0.E(r^j) \end{aligned}$$

The above formula says that in case of a risk neutral agent, the price of an asset just equals the expected present discounted value (PV) of the cash flow it generates, using the risk free interest rate for discounting.

Case 2 No Uncertainty in the economy (No aggregate risk): This is the case when the representative agent's income is constant across states;

$$w^0 = w^1 = ... = w^s$$

In this case regardless of the presence of growth, the SDF will be constant

$$\begin{aligned} M_s &= \delta.\frac{u^{'}(w^s)}{u^{'}(w^0)} = \delta = q^0 \\ &\Rightarrow \quad E(R^j) = R^0, \ \forall j \end{aligned}$$

This shows that only aggregate risk affects prices, in other words ideosyncratic risk is diversifiable if markets are complete and has no impact on asset prices.

Quadratic Utility Representative Agent and CAPM

We suppose that there is a special asset, called m, whose return is perfectly negatively correlated with the state contingent marginal utility of the representative agent, that is to say

$$\begin{aligned} R^m_s &= -au^{'}(w^s)+b, \quad \text{a>0} \\ -u^{'}(w^s) &= \frac{R^m_s - b}{a} \end{aligned}$$

If we plug in to the **CCAPM** formula we obtain

$$E(R^j) - R^0 = \frac{cov(R^m_s, R^j)}{E(u^{'}(w))}.\frac{1}{a} \quad \forall j$$

Since this holds for all assets, it must also hold for the specific asset m, hence

$$E(R^m) - R^0 = \frac{var(R^m)}{E(u^{'}(w))}.\frac{1}{a}$$

if we divide the first by the second we get a formula without marginal utility;

$$\frac{E(R^j) - R^0}{E(R^m) - R^0} = \frac{cov(R^m_s, R^j)}{var(R^m)} \quad \forall j$$

Defining

$$\begin{aligned} B_j &= \frac{cov(R^m_s, R^j)}{var(R^m)} \\ E(R^j) &= R^0 + \beta_j.[E(R^m) - R^0] \end{aligned}$$

This formula is the wellknown **Capital Asset Pricing Model (CAPM)**. To be able to use this formula there must be an asset whose return is perfectly negatively correlated with the marginal utility. Suppose **m (market portfolio)** is the claim on aggregate or mean endowment , so that $r^m_s = w^s$. Let the price of this asset q_m then we have

$$R^m_s = \frac{w^s}{q_m}.$$

We will make a further assumption, that the representative agent has a quadratic utility, i.e

$$\begin{aligned} u(w) &= cw - dw^2, \quad d > 0 \\ u^{'}(w) &= c - 2dw^s \Leftrightarrow w^s = \frac{c - u^{'}(w)}{2d} \end{aligned}$$

Then plugging in into market gross return

$$R^m_s = \frac{c - u^{'}(w)}{2dq_m} = \frac{c}{2dq_m} - \frac{u^{'}(w)}{2dq_m}, \quad 2dq_m > 0$$

so, we notice that in this special case the market return is perfectly negatively correlated which justifies the use of CAPM, which in fact is a special case of CCAPM. On the one hand CAPM is a very suitable model for empirical analysis, since it includes only observable variables, that's why it is so popular and widely used. One the other hand, the assumption of quadratic utility is too strong and not realistic and the special asset which is chosen as market portfolio is usually narrowly defined (**Roll's Critique**) either equity market or some other index which obviously does not represent the whole economic activity.

Asymmetric Information

Until now we did not deal with the information issue and assumed that there is symmetric information among agents, but this is again a very strong assumption and hence models with asymmetric information are developed to cope with such information problems. To illustrate the problem we will start with the following example.

Example Assume that we have the model with the following assumptions

- two agents: i, j
- state space : $S = 5$
- complete markets
- π := prior distribution
- 3 periods: -1:=ex-ante, 0:=interim, 1:=ex-post

We also assume that agents receive **signal** that is random variable correlated with the states of the world. Signals y(s) are such that

$$y_i(s) = \left\{ \begin{array}{ll} 1 & \text{if } \{s_1, s_2, s_3\} \\ 2 & \text{if } \{s_4, s_5\} \end{array} \right\}$$
$$y_j(s) = \left\{ \begin{array}{ll} 1 & \text{if } \{s_1, s_2\} \\ 2 & \text{if } \{s_3, s_4, s_5\} \end{array} \right\}$$

Signals can be interpreted as additional information on which state might occur; in this example if agent i receives 1 as a signal she knows that either s_1,s_2 or s_3 might occuri whereas if agent j receives the same signal he will infer that s_1 or s_2 might occur. In such model, if agent sees that an asset has positive price p_3 (i.e in state 3), she will be tempted to short sell the asset in an infinite amount, since she knows that such a state will not occur given her signal(s=1) . Therefore, in a world with asymmetric information, we can not guarantee anymore the existence of an A.D equilibrium, so we have to formalize the equilibrium concept in a new framework (**Rational Expectation models,** where agents collect information about others' information directly from prices).

Now we will formalize such an economy where there is asymmetric information; assume we have S states and I agents, then the economy is summarized by individual utilities and initial wealth: $(u_i, w_i)_{i=1}^{I}$. We will have the same assumption on utility function, namely $u^{'} > 0, u^{"} < 0$, then we will define the **posterior distribution**

$$v^s(y_i) = prob(\tilde{s} = s|\tilde{y}_i = y_i)$$

We will also assume that at ex-ante stage there will be no asymmetric information, so usual competitive equilibrium can be achieved. Then we claim that there will be no trade at interim stage given the equilibrium at t=-1, so in other words, *ex-ante* Pareto optimality implies *interim* Pareto optimality. (The reasoning is the following, the agents do not want to trade with others who want to trade with them after an equilibrium achieved.)

Definition A feasible allocation x is **ex-ante Pareto optimal** if there exist no other feasible allocation x$^{'}$ such that

$$\begin{aligned} Ex-ante\ P.O &: \sum_{s=1}^{S} \pi_s u_i(x_i^{s^{'}}) \geq \sum_{s=1}^{S} \pi_s u_i(x_i^s)\ \forall i \\ Interim\ P.O &: \sum_{s=1}^{S} v^s u_i(x_i^{s^{'}}) \geq \sum_{s=1}^{S} v^s u_i(x_i^s) \end{aligned}$$

with at least one strict inequality.

So, the **No-trade Theorem** says that, ex-ante Pareto optimality⇒interim Pareto optimality. We will mainly focus on the interim stage where there is asymmetric information.

We will consider the *standard consumption problem; there is a single good and no consumption at t=0. Then utility maximization becomes*

$$\begin{array}{c} \max_{x_i} \sum_{s=1}^{S} v^s u_i(x_i) u_i(x_i^s) \\ s.t\ \ \sum_{s=1}^{S} p^s x_i^s \leq \sum_{s=1}^{S} p^s w_i^s \end{array}$$

Recall the equilibrium defined a la A.D:

1. x^* solves UMP
2. $\sum_{s=1}^{S} x_i^s \leq \sum_{s=1}^{S} w_i^s, \forall s$

The novelty in this new model is the probabilities are replaced with posterior distributions. We know that in expected utility representation prices depend on probibility π, *so* we would expect that prices would depend on signals (posterior distribution.);

$$p^* = \phi(y_1, ...y_I)$$

but this will not be the same equilibrium prices as in A.D case. Let's assume that price functional is invertible, then the agents can see the signals and revise their probabilities

$$y = (y_1, ...y_I) \in \phi^{-1}(p^*)$$

So they will update the probabilities

$$v^s(y_i, \phi^{-1}) = prob(s = s|y_i \text{ } and \text{ } y_{-i})$$

where $y_{-i} := signals \ of \ others.$