Lecture 12 / Week 6

Green Lucas Equilibrium

Definition A Green-Lucas equilibrium is a price functional :$\phi^*() : \mathbb{R}^I \to \mathbb{R}$ such that $\sum_{i=1}^I x_i^s(y_i, p^* = \phi^*(y)) = \sum_{i=1}^I w_i^s \ \forall s$, (demand=aggregate endowment), where x_i maximizes expected utility.

We will make use of the following two mathematical results to construct our new equilibrium concept.

1. Lognormal Distribution: Let X$\sim N(\mu, \sigma^2)$. Then Y=exp(X) is lognormally distributed.

$$E(Y) = E(\exp(X)) > \exp(E(X))$$

   recall the trick that E(u(X))<u(E(X)) if u is concave, so the Jensen's Inequality is the opposite case. Then

$$E(Y) = E(\exp(X)) = \exp(\mu + \frac{1}{2}\sigma^2)$$

2. Projection Theorem: Let (X,Y) be jointly binormal, i.e.

$$\left[\begin{array}{c} X \\ Y \end{array}\right] \sim N\left(\left(\begin{array}{c} \mu_X \\ \mu_Y \end{array}\right), \left[\begin{array}{cc} \sigma_X^2 & \sigma_{XY} \\ \sigma_{XY} & \sigma_Y^2 \end{array}\right]\right)$$

Then the conditional

$$\begin{aligned} X|Y &\sim N(\mu_{X|Y}, \sigma^2_{X|Y}) \\ \mu_{X|Y} &= \mu_X + \frac{\sigma_{XY}}{\sigma_Y^2}(Y - \mu_Y) \\ \sigma^2_{X|Y} &= \sigma_X^2 - \frac{(\sigma_{XY})^2}{\sigma_y^2} \end{aligned}$$

Grossman Economy

We have the following assumptions:

- two dates t=0,1
- no consumption today

- two assets

$$\begin{aligned} \text{riskfree asset } \mathrm{r}_0 &= 1 \\ \text{risky asset } &: \widetilde{f} \sim N(\mu_f, \sigma_f^2) \end{aligned}$$

where f :fundamental value.

- agents: continuum of investors [0,1]:

$$\begin{aligned} \text{CARA utility } &: u_i(w_i) = -\exp(-\gamma_i w_i) \ \forall i \\ \text{endowments } &: \overline{x}_i \quad \forall i \\ \overline{x} &: = \int_0^1 \overline{x}_i \ di. \end{aligned}$$

- asymmetric information

$$\begin{aligned} \lambda &\in (0,1) : \text{informed agents: } \mathrm{y}_{i,I} = \widetilde{f} + \varepsilon_i, \ \varepsilon_i \sim \mathrm{N}(0, \sigma_{\varepsilon_i}^2), \varepsilon_i \perp \varepsilon_j \\ (1-\lambda) &\in (0,1) : \text{ uninformed agents: no signal} \Rightarrow y_{i,U} = \{\emptyset\} \end{aligned}$$

- we will further make no heterogeniety assumption w.l.o.g, then

$$\begin{aligned} \gamma_i &= \gamma \quad \forall i \\ y_{i,I} &= y_I \quad \forall i \text{ (same information for informed)} \\ \sigma_{\varepsilon_i}^2 &= \sigma_\varepsilon^2 \text{ (same signal)} \end{aligned}$$

Utility Maximization Problem

$$\begin{aligned} \max_{x_{i,0}, x_i} \quad E(u(w_i)) &= E(-\exp(-\gamma w_i) \ ) \\ s.t \ \ p x_i + q_0 x_{i,0} &= p.\bar{x}_i \Leftrightarrow x_{i,0} = \frac{p.(\bar{x}_i - x_i)}{q_0} \\ w_i &= \widetilde{f} \ x_i + r_0 x_{i,0} \\ w_i &= \widetilde{f} \ x_i + \frac{r_0}{q_0} p.(\bar{x}_i - x_i) \end{aligned}$$

this is the portfolio problem without the asymmetric information. We set $q_0 = 1$, i.e. we normalize the prices since only the relative prices matter. We will further assume that $\bar{x}_i = 0 \ \ \forall i$. (mathematical simplification).

In the following we willl describe the maximization problem for both types of agents, so we introduce asymmetric information, below are the UMP for uninformed and informed agent respectively

Uninformed agent

$$\begin{aligned} & \max_{x_u} \quad E(-\exp(-\gamma w_u)\ ) \\ s.t\ w_u &= (\widetilde{f}-p)x_u \end{aligned}$$

Informed agent

$$\begin{aligned} & \max_{x_I} \quad E(-\exp(-\gamma w_I)\ |y_I) \\ s.t\ w_I &= (\widetilde{f}-p)x_I \end{aligned}$$

First we solve the problem for uninformed agent given our assumptions

$$\begin{aligned} w_u &\sim N(x_u(u_f-p), x_u^2\sigma_f^2) \\ -\gamma w_u &\sim N(\gamma x_u(u_f-p), \gamma^2 x_u^2\sigma_f^2) \end{aligned}$$

Then the problem becomes

$$\begin{aligned} \max_{x_u} \quad E(-\exp(-\gamma w_u)\ ) &= \min_{x_u} \ E(\exp(-\gamma w_u)\ ) = \min_{x_u} \ \exp(\gamma x_u(u_f-p) + \frac{1}{2}\ \gamma^2 x_u^2\sigma_f^2) \\ &=^{mon.\ of\ \exp} \max_{x_u}(x_u(u_f-p) - \frac{1}{2}\ \gamma x_u^2\sigma_f^2) \end{aligned}$$

so CARA assumption with lognormal distribution result provides tractabilility, i.e. closed form sollution to the problem. Now we solve the simplified problem

$$\begin{aligned} & \max_{x_u}(x_u(u_f-p) - \frac{1}{2}\ \gamma x_u^2\sigma_f^2) \\ FOC &: \ \mu_f - p - \frac{2}{2}\gamma x_u\sigma_f^2 = 0 \Rightarrow x_u^* = \frac{\mu_f-p}{\gamma\sigma_f^2} \end{aligned}$$

this is the closed form solution for the uninformed. With some little manipulation we can show that it is a similar result we have found earlier in class:

$$\begin{aligned} x_u &= \frac{p.(\frac{u_f}{p}-1)}{p^2\gamma\frac{\sigma_f^2}{p^2}} \Rightarrow p.x_u = \frac{(\frac{u_f}{p}-1)}{\gamma\frac{\sigma_f^2}{p^2}} = \frac{E(\frac{\tilde{f}}{p})-1}{\gamma var(\frac{\tilde{f}}{p})} = \\ &= \frac{E(r)-1}{\gamma var(r)}, \quad \frac{\tilde{f}}{p} : \text{gross return on stock} \\ &= \frac{E(r)-r^0}{\gamma var(r-r^0)} \end{aligned}$$

Note that what we have calculated is exactly the same as what we have found as approximation for small risk given CARA assumption,i.e.

$$\begin{aligned} z^* &= \frac{E(r-r^0)}{var(r-r^0).A(w_0.r^0)} \\ CARA &\Rightarrow A(w_0.r^0) = \gamma. \end{aligned}$$

For the informed agent we recall our assumption and exploit the projection theorem

$$\begin{aligned}
& w_I|y_I \\
w_I|y_I &\sim N(x_I(u_{f|y_I}-p), x_I^2\sigma^2_{f|y_I}) \\
\widetilde{f} &\sim N(\mu_f, \sigma_f^2) \\
\mathrm{y}_{i,I} &= \widetilde{f} + \varepsilon_i,\ \varepsilon_i \sim \mathrm{N}(0,\sigma^2_{\varepsilon_i}), \varepsilon_i \perp \varepsilon \\
\begin{bmatrix} f \\ y_I \end{bmatrix} &\sim N\left(\begin{pmatrix} \mu_f \\ \mu_f \end{pmatrix}, \begin{bmatrix} \sigma_f^2 & \sigma_f^2 \\ \sigma_f^2 & \sigma_f^2+\sigma_\varepsilon^2 \end{bmatrix}\right) \\
\widetilde{f}|y_I &\sim N(\mu_{f|y_I}, \sigma^2_{f|y_I}) \\
\mu_{f|y_I} &= \mu_f + \frac{\sigma_f^2}{\sigma_f^2+\sigma_\varepsilon^2}(y_I-\mu_f) \\
\sigma^2_{f|y_I} &= \sigma_f^2 - \sigma_f^2.(\sigma_f^2+\sigma_\varepsilon^2)^{-1}
\end{aligned}$$

Under these assumptions we obtain the an analogous solution as before

$$x_I^* = \frac{\mu_{f|y_I}-p}{\gamma\sigma^2_{f|y_I}}$$

Then we impose the market clearing condition

$$MCC: \int_0^\lambda x_I\ di + \int_\lambda^1 x_u di = 0$$

The RHS comes from the assumption that there is no initial endowment in risky asset. LHS is the aggregate demand.

$$\begin{aligned}
u_i &= u_j \\
\lambda\frac{\mu_{f|y_I}-p}{\sigma^2_{f|y_I}} + \frac{(\mu_f-p)(1-\lambda)}{\sigma_f^2} &= 0 \\
\lambda.\sigma_f^2(\mu_{f|y_I}-p) + (1-\lambda)\sigma^2_{f|y_I}(\mu_f-p) &= 0 \\
p(\lambda\sigma_f^2 + (1-\lambda)\sigma^2_{f|y_I}) &= \lambda\sigma_f^2\mu_{f|y_I} + (1-\lambda)\sigma^2_{f|y_I}\mu_f \\
p^* &= \frac{\lambda\sigma_f^2\mu_{f|y_I} + (1-\lambda)\sigma^2_{f|y_I}\mu_f}{(\lambda\sigma_f^2 + (1-\lambda)\sigma^2_{f|y_I})} = p(y_I)
\end{aligned}$$

what we found is **FRREE**. (fullr revealing rational expectations equilibrium.) One can analyse the limiting cases as $\lambda \to 1$ or $\lambda \to 0$. We can see that the A.D equilibrium concept does not serve our purpose anymore, since uninformed agents can see the price and infer the signal of the informed ones, this in turn will change the demand of the uninformed and an equilibrium will not be possible. (Grossman-Stiglitz paradox: what information can prices carry, if noone pays for the information.)

Definition Rational expectation equilibirum (REE) is a set of trades $\{x_I(y_I,p),x_u(p)\}$ and the price functional $p(y_I)$ s.t

1. $x_I(y_I,p)$ solves

$$\max_{x_I} E(-\exp(-\gamma\ (\widetilde{f}-p)x_I)|y_I,p)$$

   $x_u(p)$ solves

$$\max_{x_u} E(-\exp(-\gamma\ (\widetilde{f}-p)x_u)|p)$$

2. M.C.C:

$$\int_0^\lambda x_I(y_I,p)\ di+\int_\lambda^1 x_u(p)\ di=0$$

One follows the following strategy to solve the REE: First conjecture a price functional and start with the competitive equilibrium, solve the UMP for both types of agents that solves MCC.

Example We conjecture the price functional

$$\begin{aligned}
p(y_I) &= a+by_I \\
\text{Uninformed} &: \max_{x_u} E(-\exp(-\gamma\ w_u)|y_I) \\
\text{Informed} &: \max_{x_I} E(-\exp(-\gamma\ w_I)|y_I) \\
x_u^* &= \frac{\mu_{f|y_I}-p}{\gamma\sigma^2_{f|y_I}}=x_I^* \\
MCC &: \frac{\mu_{f|y_I}-p}{\gamma\sigma^2_{f|y_I}}=0 \\
p^{**} &= \mu_{f|y_I}=\mu_f+\frac{\sigma_f^2}{\sigma_f^2+\sigma_\varepsilon^2}(y_f-\mu_f) \\
b &= \frac{\sigma_f^2}{\sigma_f^2+\sigma_\varepsilon^2} \\
a &= \mu_f(1-b)
\end{aligned}$$