Lecture 2 / Week 1

UNCERTAINTY

Conditionality Availability of a good depends on an exogenous event.(randomness).

Example Umbrella when it rains(London) and no rain(Barcelona) can be considered two different goods.

The concept of conditionality is modelled using **event trees**. It has 3 ingredients.

1. time: t: time period, t:0,1......T
2. $\mathcal{S}$ = The set of states of the world. s= single state, s:1,2,....S State: Rain / No rain.
3. Partition of $\mathcal{S}$: ϵ_t : a subset of $\mathcal{S}$ with some properties. These are events that can happen. ε_t in ϵ_t.

For the moment we have finite time in our models.

At t=0, there is complete uncertainty.

$\varepsilon_0 = \{1, 2, ...S\}$: **root**

$\varepsilon_T = \{\{1\}, \{2\}.....\{S\}\}$: uncertainty resolves sequentially. Over time we know what happens.

ε_o	ε_1	ε_2
We don't know	event1=$\{\{1, 2, 3\}\}$	State1
anything	event2=$\{\{4, 5, 6, 7\}\}$	State2
	event3=$\{\{8, 9\}\}$	.
	We have info that	
	we gather over time.	State9
info at t=0	t=1	t=3
$\mathcal{S} = 3$		

2-Period Model

In this section we will focus on static portfolio choices and have only two periods: t: 0,1. In the above terminology, we have two partitions: $\varepsilon_0(root), \varepsilon_1$. In this models event trees are used to describe uncertainty. $\mathcal{S}$ = the state vector and we'll define L **contingent commodities**;i.e (comodity conditioned on the state(recall umbrella rain / no rain).

Definition 1 *For every physical commodity l=1,2.....L and state $\mathcal{S}$ = 0,1......S, a unit of contingent state commodity l_s is a title to receive a physical comodity l if state s realizes.*

Then we will have the following contigent comodity vector:
$\mathbf{x} = (x_1^0, x_2^0, ...x_L^0, x_1^1, x_2^1, ...x_L^1......x_L^S) \in \mathbb{R}^{(S+1)L}$.
The endowments can be defined the same way:
$\omega = (\omega_1^0, \omega_2^0, ...\omega_L^0, \omega_1^1, \omega_2^1, ...\omega_L^1......\omega_L^S) \in \mathbb{R}^{(S+1)L}$.

At time t_0 : markets are open for trade of all state contingent commodities. We exchange payments(set prices), but the actual delivery only occurs if the state is realized,i.e. delivery depends on the state, not on the price. This set-up is similar to the one we had in the previous lecture, we can use Walrasian Equilibrium concept, but the difference is now we have more markets. ($(S+1)L$, instead of L markets. The problem the agent faces is the following:

$$\max_{x_i}\ u_i(x_i)$$
$$s.t\ \mathrm{p}^T\ (x_i - \omega_i) \leq 0$$
$$\text{where }\ \mathrm{p}^T_{(1x(S+1)L)}\ ,\ \ (x_i - \omega_i)_{(S+1)Lx\ 1}.$$

The solution would be exactly as before: $\nabla u_i(x_i^*) = \lambda p$

(a vector of $(S+1)L$ dimensions)

$$\frac{\frac{\partial u_i(x_i^*)}{\partial x_{l,i}}}{\frac{\partial u_i(x_i^*)}{\partial x_{k,i}}} = \frac{p_l}{p_k}$$

So, allocations only depend on relative prices. (Classical dichotomy)

Example t=1, W(wealth), d(damage) , s=1,2 (1= good state (W), 2= bad state(W-d)) . Agents can insure against damage paying a premium.

μ = full insurance premium

c = coverage rate(0,1): How much of damage will be covered.

The agent has to decide on coverage rate.

$c\mu$ = insurance premium.

In a good state(bad state) agents will end up with W-$c\mu$; consumption at state 1 (W- d-$c\mu + cd$; consumption at State 2). The maximization problem of the agent becomes the following:

$$\max_c u(\text{W-}c\mu,\ \text{W- d-}c\mu + cd)$$
$$\text{FOC: -}\mu u_1 + u_2(d - \mu) = 0$$
$$\frac{u_1}{u_2} = \frac{d-\mu}{\mu}$$

where u_1 = the 1. derivative of utility w.r.t consumption at State 1.

$$\frac{\mu}{d} = \frac{p_2}{p_1+p_2}.$$

This is the basic result of consumption theory(marginal rate of substitution between two commodities equals the relative price of these two goods.): the premium per damage should be equal to the relative state contingent price of one unit of consumption on the bad state (relative to the price of risk-free(state independent) consumption.) As mentioned above the analyses of **Contingent Claim Economy** is the same as the standard economy, except the number of markets, hence the concepts like Walrasian equilibrium or Pareto efficient allocation can also be applied in this set-up.

Spot Markets

Now we will have a series of financial markets, namely spot markets. There finite number of states:

$$r^{j(asset)}_{S(state)} : cash\ flow, return$$

$$\mathrm{r}^j = \begin{bmatrix} r^j_1 \\ r^j{}_2 \\ \cdot \\ \cdot \\ r^j_S \end{bmatrix}_{(Sx1)} \qquad \mathbf{r}= \begin{bmatrix} r^1_1 \ldots\ldots\ldots\ldots r^j_1 \\ \ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots \\ r^1_S \ldots\ldots\ldots\ldots r^j_S \end{bmatrix}_{(Sx\ J)} = \text{return matrix}$$

Example Risk-free return: independent of state: $\mathrm{r}^{risk-free} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}_{(Sx1)}$

The important question is in which unit the cash flows should be defined. (Numeraire). There are some conventions like average consumption(a basket) or the first good in the vector.(Mas -Colell). So, defining return depends on the choice of numaraire.

Definition 2 *Arrow Security:* $e^S = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}_{(Sx1)}$ *. This security only provides in State S one unit of purchasing power of the numeraire.*

$$\mathbf{e}= \begin{bmatrix} 1\ldots\ldots\ldots\ldots 0 \\ 0.1\ldots\ldots\ldots\ldots 0 \\ \ldots\ldots 1\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots 1\ldots\ldots \\ 0\ldots\ldots\ldots\ldots 1 \end{bmatrix}_{(Sx\ S)} = \text{security matrix,}$$

q_j = Price of asset j

$\mathbf{q}=(q_1, q_2 \ldots\ldots\ldots q_J)$

α_S = Price of Arrow security

$\alpha = (\alpha_1, \alpha_2 \ldots\ldots\ldots \alpha_S)$

Definition 3 *A **portfolio** is a collection of asset units bought and sold:* $\mathbf{z}= \begin{bmatrix} z_1 \\ z_2 \\ \cdot \\ z_j \end{bmatrix}_{(Jx1)}$.

Definition 4 ***Return on a portfolio*** *(cash-flow) is given by* $\boldsymbol{r}_{(Sx\ J)}\boldsymbol{z}_{(Jx1)} = \begin{bmatrix} \sum_{j=1}^{J} r_1^j z_j \\ \sum_{j=1}^{J} r_2^j z_j \\ . \\ \sum_{j=1}^{J} r_S^j z_j \end{bmatrix}_{(Sx1)}$

Definition 5 ***Cost of portfolio*** *is given by* $\boldsymbol{q}_{(1xJ)}\boldsymbol{z}_{(Jx1)}$*. (Note that it is a scalar.)*

Law of One Price

If we have two portfolios $\mathbf{z}_{(Jx1)}$ and $\mathbf{z}'_{(Jx1)}$, then law of one price postulates if $\mathbf{r}_{(Sx\ J)}\mathbf{z}_{(Jx1)} = \mathbf{r}_{(Sx\ J)}\mathbf{z}'_{(Jx1)} \Rightarrow \mathbf{q}_{(1xJ)}\mathbf{z}_{(Jx1)} = \mathbf{q}_{(1xJ)}\mathbf{z}'_{(Jx1)}$. In words, two assets with the same payoff vector have same prices.

An application of law of one price is Arrow securities: $\mathrm{r}^j_{(Sx1)} = e_{(Sx\ S)}r^j_{(Sx\ 1)}$. In words, a portfolio on Arrow security gives the same return as the original asset. Then the cost of the portfolio: $\mathrm{q}_j = \alpha_{(1xS)} * r^j_{(Sx1)}$.$(Decomposition)$ So, we can replicate assets using Arrow securities.

Example A financial asset that pays 1 in state 1, 3 in state 2 and 0 in state 3 has the same state contingent payoff as a portfolio of 1 state 1, 3 state 2 and 0 state 3 Arrow securities. $\mathrm{r}^j = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1\ 0\ 0 \\ 0\ 1\ 0 \\ 0\ 0\ 1 \end{bmatrix} * \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$

Risk-Neutral Probabilities

The risk neutral probability is defined as $\widetilde{\alpha}_s = \frac{\alpha_s}{\beta}$, where β is the price of risk free asset. Note that these are not probabilities in its strict sense,.i.e assigning likelihood to the states, but the components of it sum up to1.

$$\beta = \alpha_{(1xS)}r^0_{(Sx1)} = \sum_{s \in S} \alpha_s \ (r^0 = \text{risk-free asset})$$

Definition 6 $\widetilde{E}$ *is defined as the* ***expectation operato****r under risk neutral probabilities.*

Then the price of an asset j is $\mathrm{q}_j = \beta\widetilde{E}(r^j)_{(Sx1)} = \frac{\widetilde{E}(r^j)}{\rho}$, where ρ(risk free return: $\frac{1}{price} = \frac{1}{\beta}$). In arbitrage pricing theory, the aim is to find risk neutral probabilities to price assets. Note that the expected gross rate of return of ant asset, evaluated with risk-neutral probabilities , equals the risk-free rate of return. $\Rightarrow \widetilde{E}(R^j) = \rho$, where $R^j_s := \frac{r^j_s}{q_j}$.

The asset economy is defined as $\{(X_i, u_i, \omega_i)_{i=1}^I,\ \mathbf{r}\ \}$

Market Span

Market span is the set of possible asset portfolio(caracterized by the cost and return) and denoted by $\mathcal{M}$(q)= span$\begin{bmatrix} -q \\ r \end{bmatrix}$ =: $\left\{ \begin{bmatrix} -q \\ r \end{bmatrix}.z \mid z \in \mathbb{R}^J \right\}$

We also define $\alpha_+ = (1\ \alpha_1\ \alpha_2\ ....\ \ \alpha_s)$, note that it is same as α defined before with a leading 1. This vector is orthogonal to $\mathcal{M}$(q).

Proof. Consider an element of market span; x $\in \mathcal{M}$(q). $\Rightarrow x = \begin{bmatrix} -q \\ r \end{bmatrix}.z$ for some portfolio z. But then, $\alpha_+ \begin{bmatrix} -q \\ r \end{bmatrix}.z = 0$, since (-q+$\alpha$r) =0 (Law of one price).

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Decision Problem

The agent maximizes utility by choosing today's consumption (x^0) , possible consumption bundles that may be realized tomorrow (x$^1, ...$x^S) and a portfolio of securities to satisfy the budget constraint at every t and state s. This decision problem can be called the integrated consumption portfolio problem. Note that the future spot prices (p$_1, ...$p$_S$) are not observable today, but the agents have beliefs about future spot prices. (B(p$_1$),......B(p$_S$)). These beliefs are conditional on state and not subject to uncertainty.

$$\max_{x,z}\ u_i(x)$$
$$\text{s.t p}_0^T\ (x_i^0 - \omega_i^0) + q.z \leq 0\text{,where } [\text{p}_{0_{(1xL)}}^T\ (x_i^0 - \omega_i^0)_{(Lx1)}] = -\text{ saving, } q.z = \text{investment (scalar)}$$
$$\text{B(p}_s)_{(Lx1)}{}^*(x^s - \omega^s)_{(Lx1)}\text{-r}_{s_{(1xL)}}.z_{(Lx1)} \leq 0 \text{ for s=1,2....}S\text{, where}$$
$$[\text{B(p}_s)_{(1xL)}^T.(x^s - \omega^s)_{(Lx1)}] = \text{ the value of excess consumption,}$$
$$\text{r}_{s_{(1xL)}}.z_{(Lx1)} = return.(*: inner\ product).$$

The problem here is that the solution might not exist. We know that the solution exists when the objective function is continuous on a compact(closed and bounded set). The utility function is continuous by construction, by assumption the budget set is closed, but boundedness might create problem(monotonic utility, more is better)$\Rightarrow$ Arbitrage argument. (See example P.48)

Definition 7 *(q,r) contains arbitrage opportunities if there exists a portfolio z,s.t*

$$\begin{bmatrix} -q \\ r \end{bmatrix}.z \geq 0$$

This means that this portfolio gives non-negative cash-flows today or tomorrow and it is strictly positive either today or in at least one future state.

Formally, the absence of arbitrage oppotunities can be expressed as follows: the market span must not intersect with the positive orphant except the origin.

$$\mathcal{M}(q) \cap \mathbb{R}_+^{S+1} = \{0\}$$

We have proved that α_+ is orthogonal to $\mathcal{M}(q)$.

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