

Lecture 3 / Week 2

Arbitrage

Example Assume that $S = 1, J = 1$ and there is only risk free asset with return r^0 and price β . Then (q,r) arbitrage condition becomes $\begin{bmatrix} -\beta \\ 1 \end{bmatrix} z \geq 0$. There is arbitrage either if $\beta = 0$ or if $\beta < 0$. This can be seen graphically in this special case. ($\mathcal{M}(q) \cap \mathbb{R}_+^{S+1} = \{0\}$) Note that this intersection never becomes empty since we are always able to choose $z=0$.

Insert here Figure 1

Proposition Arbitrage opportunities are absent ((q,r) arbitrage free) if and only if there exists an $\alpha \gg 0$ (strictly positive Arrow prices, risk neutral probabilities) s.t $q_{(1xL)} = \alpha_{(1xS)} \cdot r_{(SxL)}$

Proof Let $y = \begin{bmatrix} -q \\ r \end{bmatrix} z \geq 0$ be arbitrage opportunity. Let $\alpha \gg 0$, then $\alpha_+ = [1 \ \alpha] \gg 0$, and hence $\alpha_+ y > 0$, but this contradicts with orthogonality condition we have proved; α_+ is orthogonal to $\mathcal{M}(q)$, in other words the multiplication of any vector of market span with α_+ should be 0. Conversely, if the market allows for arbitrage, then the arbitrage cash flow (the market span as a whole) cannot be orthogonal to any strictly positive α_+ .

Relating this proposition to the previous example, since we had only one asset, namely the risk free asset, no arbitrage condition is only consistent with positive price β .

Radner Equilibrium

Let $z = (z_i)_{i=1}^I$ be the collection of portfolios for individuals in the economy. Then the **market clearing condition for financial assets** becomes

$$\sum_{i=1}^I z_i = 0$$

since for each financial asset there should exist one buyer and one seller.

Let $\{B^i(p)\}_{i=1}^I$ be the beliefs on prices of state contingent goods tomorrow. Beliefs satisfy **conditional perfect foresight (CPF)** if

$$p_s = B^i(p_s) \text{ for } \forall i \in I, \forall s \in S$$

This condition tells us that equilibrium requires that every agent in the economy should have the same belief on the prices and these beliefs should be correct. However, this foresight is conditional on state, and there is still uncertainty about the state, so unconditional perfect foresight is not assumed.

The market clearing condition (**MCC**) on physical goods and the utility maximization problem (**UPM**) remains the same, namely

1. **MCC on physical goods:** $\sum_{i=1}^I x_i^s = \sum_{i=1}^I \omega_i^s$, for $\forall s \in S$. (Also known as **feasibility** constraint)
2. **UPM:** $\max_{x_i, z_i} u_i(x)$
s.t $p_0^T (x_i^0 - \omega_i^0) + q \cdot z \leq 0$
 $B^i(p_s)_{(Lx1)}^* (x_i^s - \omega_i^s)_{(Lx1)} - r_{s(1xL)} \cdot z_{(Lx1)} \leq 0$ for $s=1,2,\dots,S$

Definition 1 Let $\{(X_i, u_i, \omega_i)_{i=1}^I, \mathbf{r}\}$ be the asset economy.

The 4-tuple $(x_{((S+1)xL)xI}^*, P_{((S+1)xL)}^*, z_{(Lx1)}^*, q_{(1xL)}^*)$ is called Radner Equilibrium, if it fulfills the following conditions;

1. x^* solves **UPM**.
2. The **MCC**(feasibility constraint) holds.
3. z^* fulfills the market clearing condition for financial assets.
4. Beliefs satisfy the conditional perfect foresight. (**CPF**)

The problem can be split into two parts, namely **consumption composition problem(CCP)** and **financial problem(FP)**.

First notice that under monotonicity assumption and applying **CPF**, the budget constraint for UMP can be written with equality as follows

$$\begin{aligned} p_0 * (x_i^0 - \omega_i^0) &= -q \cdot z \\ p_s_{(Lx1)} * (x_i^s - \omega_i^s)_{(Lx1)} &= r_{s(1xL)} \cdot z_{(Lx1)} \end{aligned}$$

Writing the budget constraint explicitly in vector form

$$\begin{pmatrix} p_0 * (x_i^0 - \omega_i^0) \\ p_1 * (x_i^1 - \omega_i^1) \\ \vdots \\ p_S * (x_i^S - \omega_i^S) \end{pmatrix}_{(S+1)x1} = \begin{bmatrix} -q \\ r \end{bmatrix} z$$

In short notation the decision problem becomes

$$\max_{x_i, z_i} \{u_i(x_i) \mid p * (x_i - \omega_i) \in \mathcal{M}(q)\}$$

CPP

$$\begin{aligned} &\max_{x_i} u_i(x) \\ &\text{s.t } p_s x_i^s \leq y^s, \quad s=0,1,2 \dots S \end{aligned}$$

where $y=(y^0, y^1, \dots, y^S)$ is the income distribution (today and tomorrow),
 $p_s x_i^s =$ money spent on consumption.

The indirect utility is defined as follows

$$v(y) = u_i(\hat{x}_i)$$

where \hat{x}_i is the optimum consumption.

$$v(y) : = \{u(x) \mid p_s x^s \leq y^s \text{ for } s=0,1,\dots,S\}$$

$v(y)$ is the maximized utility if at most y^s can be spent in state s .

FP

$$\begin{aligned} & \max_{z_i} v(y) \\ \text{s.t } & p_0 \omega_i^0 - q z_i = y^0 \\ & p_s \omega_i^s + r_s z_i = y^s \end{aligned}$$

where $p_s \omega_i^s$ is the value of endowment and $q z_i$ is the return on portfolio.

We define $W_i^s = p_s \omega_i^s, s=0,1,2 \dots S$

then the budget constraint becomes

$$\begin{aligned} y^0 - W_i^0 &= -q z_i \\ y^s - W_i^s &= r_s z_i \end{aligned}$$

Then the problem can be written as

$$\begin{aligned} & \max_{z_i} v(y) \\ \text{s.t } & y - W_i \in \mathcal{M}(q) \end{aligned}$$

Even shortly

$$\begin{aligned} \max_{z_i} \{v(y) \mid y - W_i \in \mathcal{M}(q)\} &= \max_{z_i} \{\max_{x_i} \{u_i(x_i) \mid p_s x_i^s = y^s, \forall s \mid \\ & y - W_i \in \mathcal{M}(q)\} = \\ &= \max_{x_i, z_i} \{u_i(x_i) \mid p * (x_i - \omega_i) \in \mathcal{M}(q)\}. \end{aligned}$$

By splitting the problem into two; CCP and FP, and introducing indirect utility function $v(y)$ and state contingent value of endowment W_i^s , we can consider a new economy, 2-tuple(v,W), a contingent claim economy with only one commodity, consumption (income) in today and each future state. By using a representative commodity, we lose information on composition of consumption, but still characterize equilibrium asset prices.

Reverse Decomposition

We have seen as an application of Law of one price, that a general asset can be represented by a portfolio of Arrow securities. (Decomposition: $q_j = \alpha_{(1xS)} * r_{(Sx1)}^j$). Is the converse true? Can we generate an Arrow security with a combination of general financial assets? I.o.w is **reverse decomposition** possible?

The answer is it depends! In general we cannot say that we can always do reverse decomposition; it depends on the size and structure of the return matrix **r**.

Example If there are 5 states and only one financial asset, we cannot generate $\alpha_1, \alpha_2, \dots, \alpha_5$ simply by observing the price q_1 .

Reverse decomposition is possible if and only if financial assets provide diverse enough state-contingent cash-flows. ($J \geq S$)

Definition 2 *Markets are **complete** if agents can insure each state separately.*

If markets are complete then there is a portfolio for each state that generates the state contingent cash flows of the state- s Arrow security. Formally, for each state s there exists a portfolio z^s s.t

$$\begin{aligned} r_{(S \times J)} \cdot z_{(J \times 1)}^s &= e^s, \forall s \in S \\ r \cdot [z^1 \dots z^S] &= \mathbf{e} \end{aligned}$$

The portfolios z^s can only be computed (reverse decomposition) iff \mathbf{r} is convertible

$$[z^1 \dots z^S] = \mathbf{r}^{-1}.$$

So, markets are complete iff \mathbf{r} is convertible. Then Arrow prices can be computed from the financial market prices: $\alpha = q \cdot \mathbf{r}^{-1}$. (**Uniqueness** of Arrow prices) Mathematically, this condition to hold, the rank of the return matrix should be equal the number of states. ($J \geq S \iff \text{rank}(\mathbf{r}) = S$). If this is not the case, there are many possible Arrow prices that are compatible with return matrix \mathbf{r} and financial market prices q .