Lecture 4 / Week 2

LOTTERY

Gambles (risky situations) can be represented by a list of possible outcomes (pay-off, return) and their respective probabilities. Formally;

$$L = [x_1, \pi_{1;} x_2, \pi_2 ....x_S, \pi_S],\ \ \pi_s \geq 0 \text{ and } \sum_{s=1}^{S} \pi_s = 1$$

This is called a Lottery (L), where x, π, s represent the payoff, probability and states, respectively. Why do we need lotteries? Because they will help us to represent risky assets. Some special types of lotteries are:

Binary Lottery: $[x_1, \pi_{;} x_2, 1-\pi]$

Degenerate Lottery: $[x, 1]$.

Compound Lottery defined on L_1, L_2: $L_c = [L_1, \pi; L_2, 1-\pi] \Rightarrow we$ can construct the reduced lottery.

Let the set of all lotteries be $\mathcal{L}$. We will define preferences over this set like in ordinal utility theory with the usual assumptions. (asymetric, negatively transitive, continuous) Then we can represent such preferences with a continuous function $v : \mathcal{L} \rightarrow \mathbb{R}$. s.t $L \succeq L^{'} \Leftrightarrow v(L) \succeq v(L^{'})$.

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We will have three main assumptions to have expected utility representation over lotteries, namely,

1. **consequentialism**: People are indifferent between reduced and compound lotteries. $\Rightarrow L^R \sim L^C$
2. **state independence**: $[x, \pi_{;} y, 1-\pi] \sim [y, 1-\pi; x, \pi]$. This assumption tells us it should be the price(payoff) not the label that affects the decision.
3. **irrelavance of common alternatives**: Assume we have $L, L^{'}, L^{"} \in \mathcal{L}$ and $\alpha \in (0,1)$, then

$$L \succeq L^{'} \Leftrightarrow\ \ \alpha L + (1-\alpha)L^{"} \succeq \alpha L^{'} + (1-\alpha)L^{"}$$

So in words, the common alternatives should not change the direction of our prefence relation. Under these assumptions with the assumptions we made on preferences we can define the **expected utility representation(EUR)**:

$$U(L) = U([x_1, \pi_{1;} x_2, \pi_2 ....x_S, \pi_S]) = \sum_{s=1}^{S} \pi_s u(x_s)$$

where $u(x_s)$ is the von Neumann-Morgenstern utility. $(U(L) = E(u(L))$. In other words, von Neumann-Morgenstern utility represent EUR as a linear function of probabilities. Notice that EUR is an **ordinal** utility function, i.e. any monotonic transformation does not change the ranking thus represent the same preferences, whereas $u(x_s)$ is an **cardinal** utility function that is invariant only under positive affine transformations. (i.e $\widetilde{u} \equiv u \Leftrightarrow \exists a, \exists b > 0\ , \forall x,\ \widetilde{u}(x) = a + bu(x))$.

RISK

Definition 1 *Let L be a non-degenerate Lottery and $L^{'}$ be a degenerate lottery with $L^{'} = [E(L), 1]$. Then we can classify risk attitutes of investors as follows;*

i) investor is **risk-averse**:= $U(L^{'}) > U(L) \qquad U(E(L)^{'}) > U(L)$

ii) investor is **risk-neutral**:= $U(L^{'}) = U(L) \qquad U(E(L)^{'}) = U(L)$

iii) investor is **risk-lover**:= $U(L^{'}) < U(L) \qquad U(E(L)^{'}) < U(L)$

Insert here Figure 1(4.3,P.74, L(2004))

In this figure we have shown three lotteries: $L := [z, 1]\ z > 0,\ L^{'} := [\frac{z}{\pi_1}, \pi_1; 0, \pi_2]\ , L^{"} := [0, \pi_1; \frac{z}{\pi_2}, \pi_2]$ and showed that we have a similar curves like a budget constraint and indifference curves($p_1x_1 + p_2x_2 = w$ vs $\pi_1x_1 + \pi_2x_2 = z$)and that the gradient of the utility vector is collinear with the probabilities.

Definition 2 *The amount of money that an investor willing to pay to be indifferent between taking the sure amount of money and the lottery is called certainty equivalent amount. This phenomenon is called* ***certainty equivalence****. Formally, $U([CE(L), 1]) = U(L)$. Another description of risk aversion is, an investor is risk averse iff $E(L) > CE(L)$. The difference $RP(L) = E(L) - CE(L)$ is called* ***risk premium****. Note that positive risk premium$\Leftrightarrow$ risk aversion.*

Risk aversion can also be shown graphicallys $\Rightarrow$**concave** von Neumann-Morgenstern utility function$\Leftrightarrow$ risk aversion. $(U(E(L) \geq E(u(L)))$Formally, $|u^{"}_A(w)|>|u^{"}_B(w)|$, for $\forall w$, then investor A is more risk averse than investor B.

Insert here Figure 1(4.8,P.80, L(2004))

We have already mentioned that $u_A(x)$ is invariant only under positive affine transformations. (i.e $\widetilde{u} \equiv u \Leftrightarrow \exists a, \exists b > 0\ , \forall x,\ \widetilde{u_A}(x) = a + bu_A(x))$, but than $\widetilde{u}^{"}_A(w) = bu^{"}_A(w), b > 0$. So, concavity alone (second derivative of utility funciton) is not a proper measure for risk, since it gives different levels for the same investor. We'll normalize it by the first derivative and define:

$$\begin{aligned} ARA(Absolute\ Risk\ Aversion) &= A_u(w) = -\frac{u^{"}(w)}{u^{'}(w)} \\ RRA(\mathrm{Re}lative\ Risk\ Aversion) &= w.A_u(w) = R_u(w) \end{aligned}$$

These measures are invariant under any affine transformations. ($A_u(w) = A_{\widetilde{u}}(w), R_u(w) = R_{\widetilde{u}}(w)$). To see the usefullness of these measures we'll define the following binary lottery: $[h, \pi; -h, 1-\pi]$ and the probability $\widetilde{\pi} = \pi(h, w)$, i.e the probability that makes the investor take the lottery or not. We also define

$$u(w) = \widetilde{\pi}.u(w+h) + (1-\widetilde{\pi}).u(w-h) \qquad (*)$$

where RHS= the utility by refusing the lottery, LHS= the expected utility of accepting the lottery. Then

$$\widetilde{\pi} \simeq \frac{1}{2} + \frac{1}{4}.h.A_u(w)$$

Proof We take the following Taylor approximations around h=0 (P.60-61, DD 2005);

$$\begin{aligned} u(w+h) &= u(w) + h.u^{'}(w) + \frac{h^2}{2}.u^{"}(w) + o(h^2) \\ u(w-h) &= u(w) - h.u^{'}(w) + \frac{h^2}{2}.u^{"}(w) + o(h^2) \end{aligned}$$

We subsitiute these approximations in (*) and solve for $\widetilde{\pi}$ using $A_u(w)$ definition. This $\widetilde{\pi}$ tells us that more risk averse(i.t.o ARA) people require higher probabilities for positive return to accept the lottery.

In a similar fashion, it can also be shown for RRA, where we define $h = \theta.w$ (payoff relative to wealth) and the lottery $L = [\theta.w, \pi; (1-\theta.w), 1-\pi]$ and approximate a la Taylar around $\theta = 0$, we show that

$$\widetilde{\widetilde{\pi}} \simeq \frac{1}{2} + \frac{1}{4}.\theta.R_u(w)$$

Definition 3 *We say that the utility function exhibits constant absoulute risk aversion, CARA, if $A_u^{'}(w) = 0$,i.e. A does not depend on wealth, so the same measure would hold ,independent how rich the person is. (IARA= increasing absoulute risk aversion, DARA=decreasing absoulute risk aversion.)*

$$\begin{aligned} CARA &: A_u^{'}(w) = 0 \\ IARA &: A_u^{'}(w) > 0 \\ DARA &: A_u^{'}(w) < 0 \end{aligned}$$

Definition 4 *We say that the utility function exhibits constant relative risk aversion, CRRA, if $R_u^{'}(w) = 0$,i.e. R does not depend on wealth, so the same measure(in relative terms to initial wealth) would hold ,independent how rich the person is. (IRRA= increasing relative risk aversion, DARA=decreasing relative risk aversion.)*

$$\begin{aligned} CRRA &: \quad R_u^{'}(w) = 0 \\ IRRA &: \quad R_u^{'}(w) > 0 \\ DRRA &: \quad R_u^{'}(w) < 0 \end{aligned}$$

Empirical evidence shows that most agents have 1.) strictly increasing, 2.) strictly concave and 3.) DARA utility functions with not too large relative risk aversion. (0<R(w)<4 $\forall w$)