Lecture 5 / Week 3

STOCHASTIC DOMINANCE

Instead of evaluating the risky projects just by mean-variance analysis, a general approach has been developed using the probability distributions. We define $F_L(x) :=$ **cumulative distribution function.** Then we can compare two Lotteries L and $L^{'}$ in the following way;

Definition 1 *Let $F_L(x)$ and $F_{L'}(x)$ be defined over [0,1]. Then $F_L(x)$* ***FSD (first order stochastically dominates)*** *$F_{L'}(x)$ iff $F_L(x) \leq F_{L'}(x)$.*

Insert here Figure 1(4.5,P.68, DD(2005))

Proposition $F_L(x)$ **FSD** $F_{L'}(x) \Leftrightarrow E(u(L)) - E(u(L^{'})) \geq 0$ with u nondecreasing.

Proof

"$\Leftarrow$ " : (Proof by Contradiction, $\neg b \rightarrow \neg a$) Assume that $\exists\ \overline{x} \in$[0,1] and $F_L(x) - F_{L'}(x) > 0.(Not\ L$ FSD $L^{'}$). We define the following nondecreasing utility function $\overline{u}(x)$: (MC notation: $U(F) = \int u(x)dF(x)$ where $U() = vNM$, $u() = Bernoulli$)

$$\overline{u}(x) = \begin{Bmatrix} 1 & \overline{x} \leq x \leq 1 \\ 0 & 0 \leq x \leq \overline{x} \end{Bmatrix}$$

$$\begin{aligned}
Then E(\overline{u}(L)) - E(\overline{u}(L^{'})) &= \int_0^1 \overline{u}(x)dF_L(x) - \int_0^1 \overline{u}(x)dF_{L'}(x). \\
& \quad (Expected\ value\ represented\ by\ integrals.) \\
&= \int_0^1 \overline{u}(x)d(F_L(x) - F_{L'}(x))(Since 0 \leq x \leq \overline{x} \Rightarrow \overline{u}(x) = 0) \\
&= \int_{\overline{x}}^1 \overline{u}(x)d(F_L(x) - F_{L'}(x)) \\
Then\ we\ integrate\ by\ parts &: Recall \int udv = [u.v] - \int vdu \\
&= [\overline{u}(x).(F_L(x) - F_{L'}(x))]_{\overline{x}}^{1} - \int_{\overline{x}}^1 (F_L(x) - F_{L'}(x)).\overline{u}^{'}(x)dx \\
(Notice\ that\ F_L(1) &= F_{L'}(1) = 1\ and\ \overline{u}^{'}(x) = 0(constant\ function)) \\
&= -(F_L(\overline{x}) - F_{L'}(\overline{x})) < 0. \\
(Since\ by\ assumption\ F_L(x) - F_{L'}(x) &> 0, then\ we\ reach\ contradiction. \Rightarrow E(\overline{u}(L)) - E(\overline{u}(L^{'})) < 0)
\end{aligned}$$

"$\Rightarrow$" : W.l.o.g: Assume a differentiable $u(x)$ with $u^{'}(x) > 0$.
Note again that

$$E(\mathrm{u}(L)) - E(\mathrm{u}(L^{'})) = \int_0^1 u(x)dF_L(x) - \int_0^1 u(x)dF_{L'}(x)$$

$$= \int_0^1 u(x)d(F_L(x) - F_{L'}(x))$$

$$(Integration\ by\ parts)$$
$$= [u(x).(F_L(x) - F_{L'}(x))]_0^1 - \int_0^1 (F_L(x) - F_{L'}(x)).u^{'}(x)dx$$
$$(First\ term = 0\ as\ before)$$

$$= \int_0^1 (F_{L'}(x) - F_L(x)).u^{'}(x)dx \geq 0.$$
$$(We\ used\ FSD\ and\ the\ fact\ u^{'}(x)\ positive. QED.)$$

Insert here Figure 1

Note that it is not state by state dominance, but still extremely strong condition. Since it does not use the concept of risk aversion, one can hope for a broader measure that includes risk aversion.

Definition 2 *Let $F_L(x)$ and $F_{L'}(x)$ be two cumulative distribution functions defined over [0,1], and we have two lotteries with the same expected value, i.e. $\int_0^1 x.dF_L(x) = \int_0^1 x.dF_{L'}(x)$. Then $F_L(x)$* ***SSD (second order stochastically dominates)*** *$F_{L'}(x)$ iff $\int_0^x (F_L(t) - F_{L'}(x))dt \leq 0$ $\forall x \in$ [0,1].*

Insert here Figure 2

Notice that from the graph in case FSD the curves of cumulative distribution never cross, but this might be the case in case of SSD. Also, note that the area between curves are the same.

Proposition $F_L(x)$ **SSD** $F_{L'}(x)$ (L is less risky than $L^{'}$) iff $E(u(L)) > E(u(L^{'}))$ for u "concave, (risk averse agent)" ($u^{'} \geq 0, u^{''} < 0$).

Proof Similar to the previos one, but slightly more complicated.

Insert here Figure 3

Definition 3 *Let $f_L(x)$ and $f_{L'}(x)$ be the probability density functions of two lotteries. If $f_{L'}(x)$ can be obtained from $f_L(x)$ by removing some of the probability weight from the center of $f_L(x)$ and distributing it to the tails in such a way that the mean is unchanged, than $f_{L'}(x)$ is related to $f_L(x)$ via a* ***mean preserving spread****. In other words, the two lotteries have the same expected return, but $L^{'}$ is riskier than L. (higher variance.)*

STATIC FINANCE ECONOMY

- We define the time seperable vNM utility in the following way: u(y^0)+ $\delta.u(y)$, where δ represents the parameter for time preference(impatience). How much we value todays consumption over future consumption. Usually, $\delta \in (0,1)$, meaning that we prefer to consume now rather than later.
- We have the same states for all assets in the economy.

Before defining our problem, we should note that we make the following simplifications:

1. The expected utility is calculated in the following way: $E(u(y)) = \sum_{s=1}^{S} \pi_s.u(y^s)$
2. Today agents are provided with wealth w^0, but tomorrow there is no wealth provided in any of the states: $w^s = 0$, $s = 1, 2, ...S$.
3. We have two assets in the economy: the risk-free asset, r^0, and a risky asset r.

Then we can define the **canonical portfolio problem** in the following way:

$$\begin{aligned} & \max_{z_0, z} E(u(y)) \\ s.t\ q.\mathbf{z}(q^0.z^0 + q.z) &\leq w^0 \\ y^s &\leq w^s + r^s.\mathbf{z},\ \forall s \end{aligned}$$

Similary,in terms of value assigned to stocks,

$$\begin{aligned} \max_{z_0, z^1} \sum_{s=1}^{S} \pi_s.u(y^s) &= U(y^1, y^2....y^S) \\ s.t\ (q^0.z^0 + q.z^1) &\leq w^0 \\ y^s &\leq r^0.z^0 + r_1^s.z^1,\ \forall s = 1, 2.....S \end{aligned}$$

Then we can define values in terms of today's prices:

$$\begin{aligned} \widetilde{z}^i &= q_i.z^i \quad i = 0, 1 \\ \widetilde{r}^0 &= \frac{r^0}{q^0} \\ \widetilde{r}^s &= \frac{r^s}{q_1} \end{aligned}$$

Following we describe the same problem in values with today's prices(i.t.o tildas) assuming monotonicity;

$$\begin{aligned} & \max_{z_0,z^1}\sum_{s=1}^{S}\pi_s.u(y^s) \\ z^0+z^1 &= w^0 \Rightarrow z^0 = w^0 - z^1 \\ y^s &= r^0.z^0 + r_1^s.z^1 \Rightarrow y^s = r^0.(w^0 - z^1) + r_1^s.z^1 \\ &= r^0.w^0 + z^1(r_1^s - r^0) \\ where\ (r_1^s - r^0) &= risk\ premium \end{aligned}$$

Then the portfolio choice can be presented just in terms of money invested in risky asse:

$$\begin{aligned} \max_{z^1}\sum_{s=1}^{S}\pi_s.u(r^0.w^0 + z^1(r_1^s - r^0)) &= E(u(y)) \\ FOC &: E[u'(y^*).(r_1^s - r^0)] = 0 \\ &= \sum_{s=1}^{S}\pi_s.u'(y_s^*).(r_1^s - r^0) \end{aligned}$$

Proposition $z_1^* > 0$ (optimum amount invested in risky asset)$\Leftrightarrow E(r_1) > r^0$

$$\begin{aligned} z_1^* &> 0 \Leftrightarrow E(r_1) > r^0 \\ z_1^* &= 0 \Leftrightarrow E(r_1) = r^0 \\ z_1^* &< 0 \Leftrightarrow E(r_1) < r^0 \end{aligned}$$

Proof Define the function $f(z) = E[u'(y).(r_1^s - r^0)]$, take the first derivative: $f^{'}(z) = E[u^{''}(y).(r_1^s - r^0)^2]$ (negative slope: $(r_1^s - r^0)^2 > 0$ and $u^{''}(y) < 0$). We also have that $f(z^*) = 0 \Leftrightarrow u'(y^*) = 0$. Then

$$\begin{aligned} z^* &= 0 (\text{No risky investment} \Rightarrow f(0) = E[u'(r^0.w^0).(r_1^s - r^0)] \\ & \quad \sin ce\ (r^0.w^0) \text{ is not stochastic} \\ &= u^{'}(r^0.w^0).E[(r_1^s - r^0)] \\ u^{'}(r^0.w^0) &> 0 \Leftrightarrow E[(r_1^s - r^0) = 0 \Leftrightarrow E(r_1^s) = r^0. \end{aligned}$$

$$\begin{aligned} z^* &> 0 \Leftrightarrow f(0) > 0 \\ u'(y) &> 0 \Rightarrow E[(r_1^s - r^0) > 0 \Leftrightarrow E(r_1^s) > r^0 \\ z^* &< 0 \Leftrightarrow f(0) < 0 \\ u'(y) &> 0 \Rightarrow E[(r_1^s - r^0) < 0 \Leftrightarrow E(r_1^s) < r^0.\ \text{QED.} \end{aligned}$$

Insert here Figure 4