Lecture 6 / Week 3

Proposition $z_1^* > 0$ (optimum amount invested in risky asset)$\Leftrightarrow E(r_1) > r^0$

$$\begin{aligned} z_1^* &> 0 \Leftrightarrow E(r_1) > r^0 \\ z_1^* &= 0 \Leftrightarrow E(r_1) = r^0 \\ z_1^* &< 0 \Leftrightarrow E(r_1) < r^0 \end{aligned}$$

Recall that to prove the previous proposition we used the following function $f(z) = E[u'(y).(r_1 - r^0)]$ it takes value 0 at optimum investment, hence

$$E[u'(y^*).(r_1 - r^0)] = 0 \text{ , where } y^s = w^0.r^0 + z.(r_1^s - r^0) \quad s = 1, 2.....S$$
$$y^s = w^0.(r^0 + \tfrac{z}{w^0}.(r_1^s - r^0))$$
$$y^s = w^0.r_p^s$$
$$\tfrac{z}{w^0} = weight \text{ i.t.o ratio relative to initial wealth}$$
$$(r^0 + \tfrac{z}{w^0}.(r_1^s - r^0)) = r_p^s \text{ gross return on portfolio.}$$

Proposition The optimum investment on risky asset for a "**small risk**" can be approximated by

$$\begin{aligned} z^* &\simeq \frac{E(r_1 - r^0)}{v(r_1 - r^0).A(w^0.r^0)} \\ E(r_1 - r^0) &= \text{expected excess return} \\ v(r_1 - r^0) &= \text{variance of excess return} \\ A(w^0.r^0) &= ARA \text{ coefficient.} \end{aligned}$$

Before proceding to the proof we can state the intuition. Keep in mind that this explanation holds for small risks. The higher the risk the worst is the approximation relative to the true value. This ratio tells us, the higher excess return is expected over the risk-free rate, the more will be the agent investing on risk asset. By the same token, the higher the expected variance of the excess return, and the more risk averse is the agent, the less she will be willing to invest on risky asset.

Proof We make a Taylor approximation of the marginal utility around the risk-free wealth $(w^0.r^0)$, i.e if the agent had invested all her wealth on risk-free asset.

$$\begin{aligned}
u^{'}(y^s) &= u^{'}(w^0.r_p^s) \\
u^{'}(w^0.r_p^s) &\simeq u^{'}(w^0.r^0) + u^{''}(w^0.r^0).z.(r_1^s - r^0) + o(r_1^s - r^0) \quad /*.(r_1^s - r^0) \\
u^{'}(w^0.r_p^s).(r_1^s - r^0) &\simeq u^{'}(w^0.r^0).(r_1^s - r^0) + u^{''}(w^0.r^0).z.(r_1^s - r^0)^2 + o[(r_1^s - r^0)^2] \ \ /\text{*Exp} \\
E[u^{'}(w^0.r_p).(r_1 - r^0)] &\simeq u^{'}(w^0.r^0).E[(r_1 - r^0)] + u^{''}(w^0.r^0).z.E[(r_1 - r^0)^2] + o[E(r_1 - r^0)^2] \\
\text{using the fact } E[u'(y^*).(r_1 - r^0)] &= 0,\ E[(r_1 - r^0)^2] = v(r_1 - r^0) \text{ and } o[E(r_1 - r^0)^2] \text{ negligible} \\
0 &\simeq u^{'}(w^0.r^0).E[(r_1 - r^0)] + u^{''}(w^0.r^0).z^*.v(r_1 - r^0) \\
z^* &\simeq -\frac{u^{'}(w^0.r^0).E[(r_1 - r^0)]}{u''(w^0.r^0).v(r_1 - r^0)} \\
\sin ce\ A(w^0.r^0) &\simeq -\frac{u^{''}(w^0.r^0)}{u^{'}(w^0.r^0)} \\
z^* &\simeq \frac{E(r_1 - r^0)}{v(r_1 - r^0).A(w^0.r^0)}. \quad \text{QED.}
\end{aligned}$$

Example

Assume we have a logarithmic utility function $u(w) = \ln(w)$. We have risky(r_1) and one risk-less asset (r^0). Tomorrow we will have a return (r_1^1) on risky asset with probability π and return (r_2^1) with probability $1 - \pi$. Since we have a risk averse agent by assumption, $E(r_1) > r^0$. We will also have the following "*no arbitrage condition*". $r_1^1 > r^0 > r_1^2$. This inequality must hold, otherwise the agent exploit the return to make arbitrage. (Buying cheap and (short) selling high).

We have the following optimization problem

$$
\begin{aligned}
& \max_{z^0,\ z^1}\ \pi.\ln(y^1) + (1-\pi).\ln(y^2) \\
s.t\ z^0 + z_1 &= w^0 \Leftrightarrow z^0 = w^0 - z_1 \\
y^s &= z^0.r^0 + z_1.r_1^s \quad \text{s=1,2.....S} \\
y^s &= (\mathbf{w}^0 - \mathbf{z}_1).r^0 + z_1.r_1^s \\
y^s &= w^0.r^0 + z_1.(r_1^s - r^0) \\
& \text{ the problem becomes only a decison of risky investment} \\
& \max_{z^1}\ \pi.\ln(y^1) + (1-\pi).\ln(y^2) \\
s.t\ y^s &= w^0.r^0 + z_1.(r_1^s - r^0) \\
FOC &: \frac{\pi}{y^1}.(r_1^1 - r^0) + \frac{1-\pi}{y^2}.(r_1^2 - r^0) = 0 \Longleftrightarrow E[u'(y^*).(r_1 - r^0)] = 0 \\
\pi.y^2.(r_1^1 - r^0) &= (1-\pi).y^1.(r^0 - r_1^2) \\
y^2 &= (w^0.r^0) + z_1(r_1^2 - r^0), \quad y^1 = (w^0.r^0) + z_1(r_1^1 - r^0) \\
\pi.((w^0.r^0) + z_1(r_1^2 - r^0)).(r_1^1 - r^0) &= (1-\pi).((w^0.r^0) + z_1(r_1^1 - r^0)).(r^0 - r_1^2) \\
(w^0.r^0).[\pi.(r_1^1 - r^0) - (1-\pi)(r^0 - r_1^2)] &= 2\pi.z_1.(r_1^1 - r^0)).(r^0 - r_1^2) \\
\text{where } [\pi.(r_1^1 - r^0) - (1-\pi)(r^0 - r_1^2)] &= \pi.r_1^1 + (1-\pi).r_1^2 - r^0, \quad \boldsymbol{\pi}.\mathbf{r}_1^1 + (\mathbf{1}-\boldsymbol{\pi}).\mathbf{r}_1^2 = \mathbf{E}(\mathbf{r}) \\
(w^0.r^0).(E(r) - r^0) &= 2\pi.z_1.(r_1^1 - r^0).(r^0 - r_1^2) \\
z^* &= \frac{w^0.r^0.(E(r) - r^0)}{(r_1^1 - r^0).(r^0 - r_1^2)}
\end{aligned}
$$

The important point to note in this example is that we have a linear relationship between the investment in risky asset and initial wealth. In other words the fraction of the initial wealth invested in risky asset does not depend on initial wealth, hence we have CRRA, i.e constant relative risk aversion. This comes from the logarithmic utility function.

One might want to ask to question of how big should the expected excess return be so that the risk averse agent is induced to make risky investment. The following inequality provides the answer:

Proposition

$$E(r) - r^0 \geq w^0.A(w^0.r^0).v(r_1 - r^0)$$

This is the lower bound that induces the agent to invest all her money on risky asset.

Proof The proof is almost the same as before, we just put the restriction is that all wealth should be invested in risky asset and no short-selling constaint.($z \geq 0$)

$$\begin{aligned}
u^{'}(y^s) &= u^{'}(w^0.r_p^s) \\
\text{optimality condition} &: E[u'(y^*).(r_1 - r^0)] = 0 \\
& \quad E[u'(w^0.r^0 + z_1(r_1 - r^0)).(r_1 - r^0)] \\
\text{whole initial wealth} &: E[u'(w^0.r^0 + w^0(r_1 - r^0)).(r_1 - r^0)] \\
z &= w^0 \geq 0 \text{ (no shortselling constraint)} \\
u^{'}(w^0.r_p^s) &\simeq u^{'}(w^0.r^0) + u^{''}(w^0.r^0).z.(r_1^s - r^0) + o(r_1^s - r^0) \\
u^{'}(w^0.r_p^s).(r_1^s - r^0) &\simeq u^{'}(w^0.r^0).(r_1^s - r^0) + u^{''}(w^0.r^0).z.(r_1^s - r^0)^2 + o[(r_1^s - r^0)^2] \\
E[u^{'}(w^0.r_p).(r_1 - r^0)] &\simeq u^{'}(w^0.r^0).E[(r_1 - r^0)] + u^{''}(w^0.r^0).z.E[(r_1 - r^0)^2] + o[E(r_1 - r^0)^2] \\
0 &\simeq u^{'}(w^0.r^0).E[(r_1 - r^0)] + u^{''}(w^0.r^0).z^*.v(r_1 - r^0) \\
E(r) - r^0 &\geq w^0.A(w^0.r^0).v(r_1 - r^0)
\end{aligned}$$

Proposition Higher expected return on risky investment is equivalent to saying negative covariance with the return of risky asset and marginal utility evaluated at the optimal portfolio.

$$\begin{aligned}
E(r_1) &\geq r^0 \Leftrightarrow cov(r_1, u^{'}(w^0.r_p^*)) \leq 0 \\
E(r_1) &\leq r^0 \Leftrightarrow cov(r_1, u^{'}(w^0.r_p^*)) \geq 0
\end{aligned}$$

Proof We will prove the first line, the proof is analog for the second line. We have the following FOC for the UMP

$$E[u'(w^0.r_p^*).(r_1 - r^0)] = 0$$

Recall the formula:

$$cov(X, Y) = E(X.Y) - E(X).E(Y) \Leftrightarrow E(X.Y) = cov(X, Y) + E(X).E(Y)$$

Then we can apply this formula to the optimality condition

$$\begin{aligned}
E[u'(w^0.r_p^*).(r_1 - r^0)] &= cov(u'(w^0.r_p^*), (r_1 - r^0)) + E(u'(w^0.r_p^*).E((r_1 - r^0))) = 0 \\
&= cov(u'(w^0.r_p^*), r_1) + E(u'(w^0.r_p^*)).E((r_1 - r^0))) = 0 \\
cov(u'(w^0.r_p^*), r_1) &= -E(u'(w^0.r_p^*)).E((r_1 - r^0)))
\end{aligned}$$

Since $u'(w^0.r_p^*) > 0$ by assumption of monotonicity, the terms $cov(u'(w^0.r_p^*), r_1)$ and $E((r_1 - r^0)$. This observation completes the proof. QED.

Notice that negative covariance with return of risky asset and marginal utility implies that there is a positive covariance between the return of risky asset and wealth at $t_1(w^0.r_p^*)$. In other words, the investors demand positive risk premium for those assets that provide bad insurance. Reversing the argument, the investors might prefer risky assets over the riskless one, i.e. willing to pay a risk premium, if it provides an insurance once a bad state occurs. This refers to the second equality.

The above proposition holds only under two cases:

1. **Normal distribution**: If the return of the risky asset is normally distributed. ($r_1 \sim N$)

   Proof We have to show that if the return of the risky asset is normally disributed, then $E(r_1) \geq r^0 \Leftrightarrow cov(r_1, w^0.r_p^*) \geq 0$. We will use the following lemma in the proof;

   Stein's Lemma: Let X, Y be bivariate normal random variables. Let $g : \mathbb{R} \to \mathbb{R}$ so that $E(|g(Y)|) < \infty$. Then

   $$cov(g(Y).X) = E[g^{'}(Y)].cov(X, Y)$$

   We set $Y = w^0.r_p^*$, $X = r_1$ (Note that since $(r^0 + \frac{z}{w^0}.(r_1^s - r^0)) = r_p^s \Rightarrow r_1 \sim N \Rightarrow r_p \sim N$) and $g(Y) = u^{'}(w^0.r_p^*)$. Using the lemma

   $$cov(u^{'}(w^0.r_p^*), r_1) = E[u^{''}(w^0.r_p^*)].cov(w^0.r_p^*, r_1)$$

   Since $u^{''}(w^0.r_p^*) < 0$, by risk aversity assumption, $cov(u^{'}(w^0.r_p^*), r_1)$ and $cov(w^0.r_p^*, r_1)$ have opposite signs and this observation combined with the previous proof completes the proof. QED.

2. **Quadratic Utility**: If the agent has quadratic utility as

   $$\begin{aligned} u(w) &= \gamma_0 w - \gamma_1 w^2 \\ u^{'}(w) &= \gamma_0 - 2\gamma_1 w > 0 \\ u^{''}(w) &= -2\gamma_1 < 0,\ \gamma_1 > 0 \\ cov(r_1, \gamma_0 - 2\gamma_1 w^0.r_p^*) &= cov(r_1, -2\gamma_1 w^0.r_p^*) \\ &\quad -cov(r_1, \gamma_1 w^0.r_p^*) \end{aligned}$$

We can see that $cov(u^{'}(w^0.r_p^*), r_1)$ and $cov(w^0.r_p^*, r_1)$ have opposite signs and this observation completes the proof. QED.