Lecture 7 / Week 4

We will conduct the following analysis, how does the demand for risky asset changes with initial wealth? Do people make more investment on risk assets once they become richer? This questions can be summarized formally,

$$z^{'}(w^0) = \frac{dz^*}{dw^0}$$

Proposition Suppose that $z^*(w^0) > 0 \Leftrightarrow E(r_1) \geq r^0$. (Risk averse agent). Then

$$\begin{aligned} A^{'}(w) &< 0 \text{ (DARA)} \Leftrightarrow z^{'}(w^0) > 0 \\ A^{'}(w) &= 0 \text{ (CARA)} \Leftrightarrow z^{'}(w^0) = 0 \\ A^{'}(w) &> 0 \text{ (IARA)} \Leftrightarrow z^{'}(w^0) < 0 \end{aligned}$$

The first case tells us that the absolute risk aversion decreases with increasing wealth. In other words, the more richer one gets the more demands the risky asset.

Proof We will define the following function $f(z) = E[u^{'}(w^0.r_p).(r_1 - r^0)]$. We know from the shape of the utility function (concave) that $f^{'}(z) < 0 \Leftrightarrow u" < 0$. We also know from the optimality condition to UMP, that $f(z^*) = 0$. Since we want to make the analysis $z^{'}(w^0) = \frac{dz^*}{dw^0}$, we will use the implicit function theorem

$$\frac{dz^*}{dw^0} = -\frac{\frac{\partial f(z^*)}{\partial w^0}}{\frac{\partial f(z^*)}{\partial z^*}}$$

But since $\frac{\partial f(z^*)}{\partial z^*} < 0$, we know that $sign(\frac{dz^*}{dw^0}) = sign(\frac{\partial f(z^*)}{\partial w^0})$. So we will use LHS to make conclusion about RHS of the equation. Recall that $(r^0 + \frac{z}{w^0}.(r_1^s - r^0)) = r_p^s \; s = 1, 2...S$, so $w^0.r_p^s = (w^0.r^0 + z.(r_1^s - r^0)) \; s = 1, 2...S$. Then we calculate the following derivative

$$\frac{\partial f}{\partial w^0} = E[u^{"}(w^0.r_p).(r_1 - r^0).r_p] \quad (*)$$

We need to show that $A^{'}(w) < 0 \Leftrightarrow z^{'}(w^0) > 0$.

We will compare the marginal utility at two different wealth levels, namely, $w^0.r^0$ vs. $w^0.r_p^s$, we know by assumption that $z > 0$. There are two cases for comparison

$$\begin{aligned} Case\ 1 &: r_1^s \geq r^0 \Rightarrow w^0.r_p^s \geq w^0.r^0 \Rightarrow u'(w^0.r_p^s) \leq u^{'}(w^0.r^0) \\ Case\ 2 &: r_1^s < r^0 \Rightarrow w^0.r_p^s < w^0.r^0 \Rightarrow u'(w^0.r_p^s) > u^{'}(w^0.r^0) \end{aligned}$$

By the same token, we conduct a similar analysis, since $A^{'}(w) < 0$

$$\begin{aligned} Case\ 1 &: r_1^s \geq r^0 \Rightarrow w^0.r_p^s \geq w^0.r^0 \Rightarrow A(w^0.r_p^s) \leq A(w^0.r^0) \\ Case\ 2 &: r_1^s < r^0 \Rightarrow w^0.r_p^s < w^0.r^0 \Rightarrow A(w^0.r_p^s) > A(w^0.r^0) \end{aligned}$$

We multiply both sides with $u'(w^0.r_p^s) > 0$ and $(r_1^s - r^0)$.

Case 1 : $r_1^s \geq r^0 \Rightarrow (r_1^s - r^0) > 0 \Rightarrow u'(w^0.r_p^s).(r_1^s - r^0).A(w^0.r_p^s) \leq u'(w^0.r_p^s).(r_1^s - r^0)A(w^0.r^0)$

Using the definition of $A(w^0.r_p^s) = -\frac{u"(w^0.r_p^s)}{u'(w^0.r_p^s)}$

$$-u^{"}(w^0.r_p^s).(r_1^s - r^0) \leq u'(w^0.r_p^s).(r_1^s - r^0)A(w^0.r^0)$$

Case 2 : $r_1^s < r^0 \Rightarrow (r_1^s - r^0) < 0 \Rightarrow u'(w^0.r_p^s).(r_1^s - r^0).A(w^0.r_p^s) < u'(w^0.r_p^s).(r_1^s - r^0).A(w^0.r^0)$.

$$-u^{"}(w^0.r_p^s).(r_1^s - r^0) < u'(w^0.r_p^s).(r_1^s - r^0)A(w^0.r^0)$$

Then we take the expectation, sum up across states

$$\begin{aligned} E[-u^{"}(w^0.r_p).(r_1 - r^0)] &< E[u'(w^0.r_p).(r_1 - r^0)A(w^0.r^0)] \\ E[-u^{"}(w^0.r_p).(r_1 - r^0)] &< A(w^0.r^0).E[u'(w^0.r_p).(r_1 - r^0)] \end{aligned}$$

Once we evaluate at the optimum $E[u'(w^0.r_p^*).(r_1 - r^0)] = 0$. Hence

$$\begin{aligned} E[-u^{"}(w^0.r_p).(r_1 - r^0)] &< 0 \\ E[u^{"}(w^0.r_p).(r_1 - r^0)] &> 0. \end{aligned}$$

This completes the proof, once we see the relation to (*). Since by assumption r$_p > 0$, then $E[u^{"}(w^0.r_p).(r_1 - r^0)] \Leftrightarrow \frac{\partial f}{\partial w^0} > 0 \Leftrightarrow A^{'}(w) < 0$. QED.

We might also want to answer the question of the weight: $\frac{\textit{amount}\text{ invested in risky asset}}{\textit{initial}\text{ wealth}}$ changes once the initial wealth changes. This question is related to the relative risk aversion. In other words, what is the percentage change in investment in risky asset once the initial wealth changes in percentage terms. (elasticity). Does people invest a higher fraction of their wealth in risky asset once they become richer? Formally we conduct the following analysis

$$\eta^* = \frac{dz^*}{dw^0}.\frac{w^0}{z^*}$$

Proposition Let $R(w)$ be the **relative risk aversion** coefficient, then

$$\begin{aligned} R^{'}(w) &< 0 \text{ (DRRA)} \Leftrightarrow \eta(w^0) > 1 \Leftrightarrow \text{the risky asset is normal good} \\ R^{'}(w) &= 0 \text{ (CRRA)} \Leftrightarrow \eta(w^0) = 1 \\ R^{'}(w) &> 0 \text{ (IRRA)} \Leftrightarrow \eta(w^0) < 1 \Leftrightarrow \text{ the risky asset is inferior good} \end{aligned}$$

We will not proof this proposition, but the interpretation is important; if your wealth changes **in percentage terms**, DRRA implies that the agent wants to invest more in risky asset, so in other words the risky asset is a normal good. (Analogly, in the last case, it is inferior good.) This analysis is the most we can conduct with the canonical portfolio problem. From now on, we will analyse cases, where we have more than one risky asset. Then, we will need another framework to see the relationship between the risk premia of different risky assets and its effect on portfolio choice.

Example The following function is DARA (CRRA)

$$\begin{aligned} u(w) &= \frac{1}{1-\gamma} w^{1-\gamma} \\ u^{'}(w) &= w^{-\gamma} \\ u''(w) &= -\gamma . w^{-(\gamma+1)} \\ A(w) &= -\frac{u^{''}(w)}{u(w)} = \frac{\gamma . w^{-(\gamma+1)}}{w^{-\gamma}} = \frac{\gamma}{w} \\ A^{'}(w) &= -\frac{\gamma}{w^2} < 0 \ (DARA) \\ R(w) &= \gamma \text{ (CRRA)} \end{aligned}$$

Modern Portfolio Theory (Mean -Variance Analysis)

We will start with a simple model where we have quadratic utility and then normal returns on portfolio.

Case 1 Quadratic Utility: We have the following utility function : $u(w) = \gamma_0 . w - \gamma_1 . w^2$. Now there are n assets in the portfolio and we consider the following lottery:

$$x = [x_1, \pi_1; ....x_s, \pi_s]$$

The expected utility becomes

$$E[u(x)] = \sum_{s=1}^{S} \pi_s . u(x_s)$$

We substitute our quadratic utility

$$\begin{aligned} E[u(x)] &= \sum_{s=1}^{S} \pi_s . (\gamma_0 . x_s - \gamma_1 . x_s^2) = \\ &= \gamma_0 . \sum_{s=1}^{S} \pi_s . x_s - \gamma_1 . \sum_{s=1}^{S} \pi_s . (x_s^2) = \\ &= \gamma_0 . E(X) - \gamma_1 . var(X) - \gamma_1 . (E(X))^2 \end{aligned}$$

Recall that $var(X) = E(X^2) - (E(X))^2 \Leftrightarrow E(X^2) = var(X) + (E(X))^2$.

To see how the expected utility changes:

$$\frac{d(E(u(x)))}{d(E(X))} = \gamma_0 - 2\gamma_1.E(X) \Longleftrightarrow u^{'} = \gamma_0 - 2\gamma_1 w > 0$$
$$\frac{d(E(u(x)))}{d(var(X))} = -\gamma_1 < 0$$

We see that $\uparrow E(X) \Rightarrow \uparrow E[u(x)]$ and $\uparrow \sigma(X) \Rightarrow \downarrow E[u(x)]$, so

$$E[u(x)] = f(E(X), \sigma(X)).$$
$$E[u(x)] = f(\uparrow, \downarrow)$$

Insert here Figure 1