Lecture 8 / Week 4

Modern Portfolio Theory (Mean -Variance Analysis)

We will see that mean-variance analysis can be successfully conducted in three cases.

1. where we have **quadratic utility**.
2. when returns of the portfolio have **jointly normal distribution**
3. When we approximate in the neighbourhood of $E(X)$.

Case 1 Quadratic Utility: We have the following utility function : $u(w) = \gamma_0.w - \gamma_1.w^2$. Now there are n assets in the portfolio and we consider the following lottery:

$$x = [x_1, \pi_1; ....x_s, \pi_s]$$

The expected utility becomes

$$E[u(x)] = \sum_{s=1}^{S} \pi_s.u(x_s)$$

We substitute our quadratic utility

$$\begin{aligned} E[u(x)] &= \sum_{s=1}^{S} \pi_s.(\gamma_0.x_s - \gamma_1.x_s^2) = \\ &= \gamma_0.\sum_{s=1}^{S} \pi_s.x_s - \gamma_1.\sum_{s=1}^{S} \pi_s.(x_s^2) = \\ &= \gamma_0.E(X) - \gamma_1.var(X) - \gamma_1.(E(X))^2 \end{aligned}$$

Recall that $var(X) = E(X^2) - (E(X))^2 \Leftrightarrow E(X^2) = var(X) + (E(X))^2$.
To see how the expected utility changes:

$$\begin{aligned} \frac{d(E(u(x)))}{d(E(X))} &= \gamma_0 - 2\gamma_1.E(X) \Longleftrightarrow u^{'} = \gamma_0 - 2\gamma_1 w > 0 \\ \frac{d(E(u(x)))}{d(var(X))} &= -\gamma_1 < 0 \end{aligned}$$

We see that $\uparrow E(X) \Rightarrow \uparrow E[u(x)]$ and $\uparrow \sigma(X) \Rightarrow \downarrow E[u(x)]$, so

$$\begin{aligned} E[u(x)] &= f(E(X), \sigma(X)). \\ E[u(x)] &= f(\uparrow, \downarrow) \end{aligned}$$

Insert here Figure 1

Case 2 Jointly normal distributed returns: We assume that we have jointly normal returns $X \sim N(\mu, \sigma^2)$. We can standardise the returns $X = E(X) + \sigma(X).z$, where $z \sim N(0,1)$. Then we can write the expected value of the utility as follows

$$\begin{aligned} E[u(X)] &= E[u(E(X)) + \sigma(X).z] \\ \frac{d(E(u(X)))}{d(E(X))} &= E[u^{'}(X)] > 0 \\ \frac{d(E(u(X)))}{d(\sigma(X))} &= E[u^{'}(X).z] < 0 \end{aligned}$$

The last equality follows from both the concavity of the utility function and by the fact that z is symmetric. (negative components are bigger.)

Insert here Figure 2

Case 3 Mean Variance Analysis as Approximation: We assume that $u(X)$ is well defined, i.e. it has finite moments in the neighbourhood of E(X). Then we can have the following Taylor expansion:

$$\begin{aligned} u(X) &\simeq u(E(X)) + u^{'}(E(X)).(X - E(X)) + \frac{1}{2}.u^{"}(E(X)).(X - E(X))^2 \\ We\ take\ \exp ectation &: \\ E(u(X)) &\simeq u(E(X)) + u'(E(X)).(E(X) - E(X)) + \frac{1}{2}.u^{"}(E(X)).\sigma^2(X) \\ \frac{d(E(u(X)))}{d(E(X))} &= u^{'}(E(X)) + \frac{1}{2}.u^{'''}(E(X)).\sigma^2(X) \quad > 0 \\ u^{'''} &: curvature\ of\ m\arg inal\ utility,\ u^{'''} > 0 \rightarrow sufficient\ condition \\ \frac{d(E(u(X)))}{d(\sigma(X))} &= \frac{1}{2}.u^{"}(E(X)).2.\sigma(X) \ < 0 \end{aligned}$$

Note that once we have quadratic utility $u^{'''} = 0$, then the condition is automatically satisfied.

Insert here Figure 3

Reduced Form of Preferences

Since we know that the expected utility is a inreasing function in expected value of returns and a decreasing one in variance of returns we will use the following simplified utility function to conduct mean variance analysis

$$E(X) - \alpha\sigma^2(X),\ \alpha > 0$$

We hope to get closed form solution for asset demands. We wnat to be able to write the budget constraint and to rank the investment opportunities according to some criterion, which will be the following

Definition Assuming that the agent has investment opportunities with the same expected return, i.e. $E(X_1) = E(X_2)$, the first one dominates the second one i.t.o mean variance, i.e. X_1 **mean variance dominates** X_2 if $\sigma(X_1) < \sigma(X_2)$. This is called **mean-variance criterion. (M-V criterion)**

Our next step will be to find the budget constraint that satisfies the M-V criterion, once the agent has a portfolio with n assets. Then

$$q_1.z_1 + q_2.z_2 + ...q_n.z_n = w^0$$

We divide by initial wealth since our focus in our analysis is about portfolio choice, i.e. how the agent split its wealth among different portfolio assets. (Before we did the analysis for the change w.r.t. initial wealth.)

$$\frac{q_1.z_1}{w^0} + ...\frac{q_n.z_n}{w^0} = 1$$

Each term in the summation is called **portfolio weight** that are normalized by initial wealth. We change the notation to $w_1 = \frac{q_1.z_1}{w^0}$. Then

$$w_1 + w_2 + ...w_n = 1$$

is socalled the fully invested portfolio. Then the net portfolio return will be

$$r_p = w_1.r_1 + w_2.r_2....... + w_n.r_n$$

Matrix Notation

portfolio weights: $\mathbf{w}=\begin{bmatrix} w_1 \\ w_2 \\ . \\ . \\ w_n \end{bmatrix}_{n\times 1}$ $\quad$ *var-cov matrix* : $\mathbf{v}=\begin{bmatrix} \sigma_1^2 & \sigma_{12} & . & \sigma_{1n} \\ & \sigma_2^2 & & \\ & & . & \\ & & & \sigma_n^2 \end{bmatrix}_{n\times n}$

exp*ected return* : $\boldsymbol{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ . \\ . \\ \mu_n \end{bmatrix}_{n\times 1}$ $\quad$ *unit vector* $\mathbf{i}=\begin{bmatrix} 1 \\ 1 \\ . \\ . \\ 1 \end{bmatrix}_{n\times 1}$

Portfolio Choice

Case 1 n risky assets, no risk-free rate

Before writing the optimization problem we will express our terms in matrix form:

$$\begin{aligned}
\mu_p &: = E(r_p) = w_1.\mu_1 + w_2.\mu_2.....w_n.\mu_n = \boxed{\mathbf{w}'.\boldsymbol{\mu}}_{(1\times n)\times(n\times 1)} \\
\sigma_p^2 &: = var(r_p) = \boxed{\mathbf{w}'.\mathbf{v}.\mathbf{w}}_{(1\times n)\times(n\times n)\times(n\times 1)} \\
\textbf{Budget constraint} &: \mathbf{w}'.\mathbf{i} = 1
\end{aligned}$$

Then the optimization problem becomes

$$\min_{\mathbf{w}} \mathbf{w}'.\mathbf{v}.\mathbf{w}$$
$$\begin{aligned}
s.t\ \mathbf{w}'.\boldsymbol{\mu} &= \mu_T \text{ (target return)} \\
\mathbf{w}'.\mathbf{i} &= 1 \text{ (Budget constraint)}
\end{aligned}$$

So we have constructed an optimization problem i.t.o. mean variance criterion. The agent wants to reach a target return that minimizes the variance of its portfolio and satisfies at the same time the budget constraint.

Example Two risky asset case:

$$\begin{aligned}
\min_{\mathbf{w}} \sigma_p^2 &= w_1^2.\sigma_1^2 + w_2^2.\sigma_2^2 + 2.w_1.w_2.\sigma_{12} \\
s.t\ \ w_1.\mu_1 + w_2.\mu_2 &= \mu_T \\
w_1 + w_2 &= 1 \Leftrightarrow w_2 = 1 - w
\end{aligned}$$

Then we can write the optimization problem only with one control variable

$$\begin{aligned}
\min_{w_1} \sigma_p^2 &= w_1^2.\sigma_1^2 + (1-w_1)^2.\sigma_2^2 + 2.w_1.(1-w_1).\varphi.\sigma_1.\sigma_2 \\
where\ cov(r_1, r_2) &= \varphi.\sigma_1.\sigma_2 = \sigma_{12} \\
s.t\ \ w_1.\mu_1 + (1-w_1).\mu_2 &= \mu_T
\end{aligned}$$

We will analyse special cases

Ex. Case 1 $\varphi = 1$: perfect positive correlation between returns of two risky asset , then we have

$$\begin{aligned}
\sigma_p^2 &= [w_1.\sigma_1 + (1-w_1).\sigma_2]^2 \Leftrightarrow \sigma_p = w_1(\sigma_1 - \sigma_2) + \sigma_2 \Leftrightarrow w_1 = \frac{\sigma_p - \sigma_2}{\sigma_1 - \sigma_2} \\
\mu_T &= w_1.\mu_1 + (1-w_1).\mu_2 \Leftrightarrow \mu_T = w_1.(\mu_1 - \mu_2) + \mu_2 = (\frac{\sigma_p - \sigma_2}{\sigma_1 - \sigma_2}).(\mu_1 - \mu_2) + \mu_2
\end{aligned}$$

If we want to see the relationship between portfolio standard deviation σ_p (x-axis) and the portfolio return μ_p (y-axis) graphically, we obtain a straight line. This curve, which is a line in this special case will be called **portfolio**

frontier. This shows us the linear relationship between the return and risk (i.t.o standard deviation) of the portfolio in this special case.

Ex. Case 2 $\varphi = -1$: perfect negative correlation between returns of two risky asset , then we have

$$\sigma_p = \left\{ \begin{array}{c} w_1.\sigma_1 - (1-w_1).\sigma_2 \\ -w_1.\sigma_1 + (1-w_1).\sigma_2 \end{array} \right\}$$

Then we have two line one with positive and one with negative slope intersecting at y-axis.(**risk-free portfolio**). Then the the portfolios on the line with with positive slope(portfolio frontier) are more effcient (M-V dominate) than the ones on the negative line.

Between these two extreme case, i.e. $-1 < \varphi < 1$, we will have a curve that lies between these two extreme cases, where we can find the minimum variance portfolio. (Only the second special case it is 0.)

Insert here Figure 4

Now we will proceed with how to solve such an optimization problem with n risky asset without a rikless one. We will set-up the Lagrangian and make use of matrix algebra.

Recall the problem

$$\min_{\mathbf{w}} \frac{1}{2}\mathbf{w}^{'}.\mathbf{v}.\mathbf{w}$$

$$\begin{aligned} s.t\ \mathbf{w}^{'}.\boldsymbol{\mu} &= \mu_T \\ \mathbf{w}^{'}.\mathbf{i} &= 1 \end{aligned}$$

Note that $\frac{1}{2}$ just brings computational simplicity and does not change the nature of the optimization problem. Recall also from matrix algebra

$$\begin{aligned} \frac{d(a^{'}.b)}{da} &= b \\ \frac{d(a^{'}.A.a)}{da} &= 2.A.a \end{aligned}$$

Then

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}\mathbf{w}^{'}.\mathbf{v}.\mathbf{w}+\lambda.(\mu_T - \mathbf{w}^{'}.\boldsymbol{\mu})+\gamma.(1-\mathbf{w}^{'}.\mathbf{i}) \\ FOC &: \frac{\partial \mathcal{L}}{\partial \mathbf{w}} = \frac{1}{2}.2.\mathbf{v}.\mathbf{w}-\lambda.\boldsymbol{\mu}-\gamma.\mathbf{i} = \mathbf{0} \qquad (n \times 1) \\ \frac{\partial \mathcal{L}}{\partial \boldsymbol{\lambda}} &= \mu_T - \mathbf{w}^{'}.\boldsymbol{\mu} = 0 \\ \frac{\partial \mathcal{L}}{\partial \boldsymbol{\gamma}} &= 1 - \mathbf{w}^{'}.\mathbf{i} = 0 \end{aligned}$$

Using matrix algebra and the fact that $\mathbf{v}$ is invertible (square matrix), we can write n+2 optimality conditions in the following way:

$$
\begin{aligned}
\mathbf{w} &= \mathbf{v}^{-1}.(\lambda.\boldsymbol{\mu}+\gamma.\mathbf{i}) \quad \text{(n conditions)} \\
& \text{we premultiply with } \boldsymbol{\mu}' \\
\boldsymbol{\mu}'.\mathbf{w} &= \boldsymbol{\mu}'.\mathbf{v}^{-1}.(\lambda.\boldsymbol{\mu}+\gamma.\mathbf{i}) = \mu_T \\
& \sin ce\ \lambda \text{ and } \gamma \text{ are scalars} \\
\boldsymbol{\mu}'.\mathbf{w} &= (\boldsymbol{\mu}'.\mathbf{v}^{-1}.\boldsymbol{\mu}).\lambda + (\boldsymbol{\mu}'.\mathbf{v}^{-1}.\mathbf{i}).\boldsymbol{\gamma} = \mu_T \\
& \text{similarly we premultiply with } \mathbf{i} \\
\mathbf{i}.\mathbf{w} &= \mathbf{i}.\mathbf{v}^{-1}.(\lambda.\boldsymbol{\mu}+\gamma.\mathbf{i}) = 1 \\
\mathbf{i}.\mathbf{w} &= (\mathbf{i}.\mathbf{v}^{-1}.\boldsymbol{\mu}).\lambda + (\mathbf{i}.\mathbf{v}^{-1}.\mathbf{i}).\boldsymbol{\gamma} = 1
\end{aligned}
$$

Then we define

$$
\begin{aligned}
A &:= \mathbf{i}.\mathbf{v}^{-1}.\boldsymbol{\mu} \\
B &:= \boldsymbol{\mu}'.\mathbf{v}^{-1}.\boldsymbol{\mu} \\
C &:= \mathbf{i}.\mathbf{v}^{-1}.\mathbf{i} \\
D &:= B.C - A^2
\end{aligned}
$$

Note that these are all scalars and we obtain

$$
\begin{aligned}
\lambda.B + \gamma.A &= \mu_T \\
\lambda.A + \gamma.C &= 1
\end{aligned}
$$

We express the lagrange multiplier as

$$
\begin{aligned}
\lambda &= \frac{C.\mu_T - A}{D} \\
\gamma &= \frac{B - A.\mu_T}{D}
\end{aligned}
$$

We plug them into the first n optimality constraints, then $\mathbf{w} = \mathbf{v}^{-1}.(\lambda.\boldsymbol{\mu}+\gamma.\mathbf{i})$ becomes

$$
\begin{aligned}
\mathbf{w} &= \mathbf{v}^{-1}.(\boldsymbol{\mu}.\frac{C.\mu_T - A}{D} + \mathbf{i}.\frac{B - A.\mu_T}{D})_{(n\times 1)} \\
&= \frac{1}{D}.\{(C.\mu_T).\mathbf{v}^{-1}.\boldsymbol{\mu} - A.\mathbf{v}^{-1}.\boldsymbol{\mu} + B.\mathbf{v}^{-1}.\mathbf{i} - (A.\mu_T).\mathbf{v}^{-1}.\mathbf{i}\} \\
& \text{We seperate the terms that depend on target return} \\
&= \frac{1}{D}.\{\mu_T.(C.\mathbf{v}^{-1}.\boldsymbol{\mu} - A.\mathbf{v}^{-1}.\mathbf{i}) + B.\mathbf{v}^{-1}.\mathbf{i} - A.\mathbf{v}^{-1}.\boldsymbol{\mu}\} \\
&= \mathbf{h}.\mu_T + \mathbf{g} \\
\mathbf{h}_{(n\times 1)} &:= \frac{(C.\mathbf{v}^{-1}.\boldsymbol{\mu} - A.\mathbf{v}^{-1}.\mathbf{i})}{D}, \ \mathbf{g}_{(n\times 1)} := \frac{B.\mathbf{v}^{-1}.\mathbf{i} - A.\mathbf{v}^{-1}.\boldsymbol{\mu}}{D}
\end{aligned}
$$

Two special cases (no expected return and 100% expected return) :

$$\begin{aligned}\mu_T &= 0 \Leftrightarrow \mathbf{w} = \mathbf{g} \\ \mu_T &= 1 \Leftrightarrow \mathbf{w} = \mathbf{g} + \mathbf{h}\end{aligned}$$

Lemma (First Seperation Theorem) The vectors g and $g+h$ span the whole frontier. So, only two portfolios are need to generate the whole frontier.

$\Rightarrow \mathbf{w}_q = \mathbf{g} + \mathbf{h}.\mu_q$, where μ_q is the target return of portfolio q. Note that $\mathbf{w}_q$ is linear combination of $\mathbf{g}$ and $\mathbf{g}+\mathbf{h}$:

$$\mathbf{w}_q = (1-\mu_q).\mathbf{g} + \mu_q.(\mathbf{g}+\mathbf{h}) = \mathbf{g} + \mathbf{h}.\mu_q$$

Then the portfolio choice becomes minimizing

$$\sigma_p^2 = \mathbf{w}^{'}.\mathbf{v}.\mathbf{w} = (\mathbf{g} + \mathbf{h}.\mu_q)^{'}.\mathbf{v}.(\mathbf{g} + \mathbf{h}.\mu_q)$$

The graph can be drawn using

$$\frac{\sigma_p^2}{1/C} - \frac{(\mu_T - A/C)}{D/C^2} = 1$$

Recall that the general equation for hyperbole is

$$\frac{(x-x_0)^2}{a^2} - \frac{(y-y_0)}{b^2} = 1$$

Insert here Figure 5

Looking at the graph, we see that the two asymtotes are $\mu_p = \frac{A}{C} \pm \sqrt{\frac{D}{C}}\sigma_p$ and the minimum variance portfolio is at $(0, \frac{A}{C})$