Lecture 9 / Week 5

Modern Portfolio Theory (Mean -Variance Analysis)

We will introduce a more general proposition than the previous lemma.

Proposition Any portfolio frontier can be generated using two (other not necessarily g and $g+h$) frontier portfolios

Proof Let p_1 and p_2 be two frontier portfolios. $[(w_{p_1}, w_{p_2}),(\mu_{p_1}, \mu_{p_2})]$ We want to show that we can replicate any frontier portfolio $q.(w_q, \mu_q)$, formally we need to show

$$\mu_q = \alpha\mu_{p_1} + (1-\alpha)\mu_{p_2}$$

We showed in the previous lemma that using g and $g+h$ we can replicate $w_p = g + h\mu_p$. Using this result

$$\begin{aligned} w_q &= \alpha w_{p_1} + (1-\alpha)w_{p_2} = \alpha(g + h\mu_{p_1}) + (1-\alpha)(g + h\mu_{p_2}) = \\ &= g + h(\alpha\mu_{p_1} + (1-\alpha)\mu_{p_2}) \\ &= g + h\mu_q. \quad QED. \end{aligned}$$

Case 1: one risky and one riskless asset.

Next we will analyze the case where **one risky and one riskless asset.** We will denote w: the portfolio weight in risky asset. But then since it is a two asset case

$$\sigma_p^2 = w^2.\sigma^2 + (1-w)^2.\sigma_{rf}^2 + 2.w_1.(1-w).\sigma_{1rf}$$

but since we have a riskfree asset, the second and third term in the summation will be 0. Hence

$$\sigma_p^2 = w^2.\sigma^2 \Rightarrow w = \frac{\sigma_p}{\sigma}$$

Then we can plug in this result in the portfolio return

$$\begin{aligned} \mu_p &= w.\mu_1 + (1-w).r^0 \\ &= w(\mu_1 - r^0) + r^0 = \frac{\sigma_p}{\sigma}(\mu_1 - r^0) + r^0 \end{aligned}$$

we can see that what we obtain a portfolio frontier which is a straight line.

Insert here Figure 1

Case 2: two risky and one riskless asset.

If we have two risky asset as presented in the following figure, where the line $(r^0 r^2)$ is above $(r^0.r^1)$, then the optimizing agent should invest her money in riskless asset and second asset, since the portfolio frontier is tangent to the indifference curve at a higher level.

Insert here Figure 2

Case 3: **n risky and one riskless asset.**

The above analysis can be extended to n risky asset case with a riskless one. Then the optimizing agent would invest money on riskfree asset and **tangency portfolio**. The composition of how much money should be spent on riskfree asset and tangency portfolio (consisting of risky assets) depend on the risk aversion of the agent which represented by parameter α_i in the reduced form of the preferences.$(E(X) - \alpha_i \sigma^2(X))$ Given two agents A and B, if $\alpha_B > \alpha_A$, then the agent B is more risk averse (steeper indifference curve which will intersect portfolio frontier between r^0 and tangent portfolio→ she is lending.) This can be seen in the below figure.

Insert here Figure 3

In this third case we will denote by w^0 :*portfolio weight in riskfree asset.* Then we can transform our previous optimization problem (with matrix notation) in the following way:

$$Before \text{ Budget contraint} : (\mathbf{w}'.\mathbf{i}=1) \Rightarrow w^0 + (\mathbf{w}'.\mathbf{i})=1 \Rightarrow w^0 = 1 - \mathbf{w}'.\mathbf{i}$$

Recall that the objective function is the variance of the portfolio, that the agent tries to minimize, but riskfree asset does not have any effect on the varience, rather it changes the constraint.

$$Before : \mathbf{w}'.\boldsymbol{\mu}=\mu_T \ \Rightarrow w^0.r^0 + \mathbf{w}'.\boldsymbol{\mu}=\mu_T$$

Plugging in the previous result

$$\begin{aligned} (1 - \mathbf{w}'.\mathbf{i})r^0 + \mathbf{w}'.\boldsymbol{\mu} &= \mu_T \\ r^0 + \mathbf{w}'.(\boldsymbol{\mu} - r^0.\mathbf{i})_{n\times 1} &= \mu_T \end{aligned}$$

Then the optimization problem becomes

$$\begin{aligned} &\min_{\mathbf{w}} \frac{1}{2}\mathbf{w}'.\mathbf{v}.\mathbf{w} \\ s.t \quad \mu_T &= r^0 + \mathbf{w}'.(\boldsymbol{\mu} - r^0.\mathbf{i})_{n\times 1} \end{aligned}$$

We set up the Lagrangian

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}\mathbf{w}'.\mathbf{v}.\mathbf{w} + \lambda.(\mu_T - r^0 - \mathbf{w}'.(\boldsymbol{\mu} - r^0.\mathbf{i})) \\ FOC &: \frac{\partial \mathcal{L}}{\partial \mathbf{w}} = \mathbf{v}.\mathbf{w} - \lambda.(\boldsymbol{\mu} - r^0.\mathbf{i}) = \mathbf{0}_{(n\times 1)} \\ \frac{\partial \mathcal{L}}{\partial \lambda} &= \mu_T - r^0 - \mathbf{w}'.(\boldsymbol{\mu} - r^0.\mathbf{i}) \end{aligned}$$

We follow the same strategy as before, first we will isolate the control variable and then get rid off the lagrange multiplier, and plugging in to the constraints we will obtain the portfolio weights, return and variance;

$$\begin{aligned}
\mathbf{w} &= \lambda.\mathbf{v}^{-1}.(\boldsymbol{\mu}-r^0.\mathbf{i}) \\
& \quad \textit{multiply by } (\boldsymbol{\mu}-r^0.\mathbf{i})' \\
(\boldsymbol{\mu}-r^0.\mathbf{i})'\mathbf{w} &= \lambda.(\boldsymbol{\mu}-r^0.\mathbf{i})'\mathbf{v}^{-1}.(\boldsymbol{\mu}-r^0.\mathbf{i}) \\
& \quad \text{Using optimality condition} \\
\lambda.(\boldsymbol{\mu}-r^0.\mathbf{i})'\mathbf{v}^{-1}.(\boldsymbol{\mu}-r^0.\mathbf{i}) &= \mu_T - r^0 \\
\text{we call } (\boldsymbol{\mu}-r^0.\mathbf{i})'\mathbf{v}^{-1}.(\boldsymbol{\mu}-r^0.\mathbf{i}) &= H \\
\lambda.H &= \mu_T - r^0 \\
\lambda &= \frac{\mu_T - r^0}{H}
\end{aligned}$$

We plug in λ to the optimal portfolio

$$\begin{aligned}
\mathbf{w} &= \frac{\mu_T - r^0}{H}.\mathbf{v}^{-1}.(\boldsymbol{\mu}-r^0.\mathbf{i}) \\
\text{since } \sigma_p^2 &= \mathbf{w}'.\mathbf{v}.\mathbf{w} \\
&= \frac{(\mu_T - r^0)^2}{H^2}.(\boldsymbol{\mu}-r^0.\mathbf{i})'.\mathbf{v}^{-1}.\mathbf{v}.\mathbf{v}^{-1}(\boldsymbol{\mu}-r^0.\mathbf{i}) \\
\text{using } H &= (\boldsymbol{\mu}-r^0.\mathbf{i})'\mathbf{v}^{-1}.(\boldsymbol{\mu}-r^0.\mathbf{i}) \\
\sigma_p^2 &= \frac{(\mu_T - r^0)^2}{H} \Rightarrow \mu_T = \sqrt{H}.\sigma_p + r^0 \\
\textit{where } \mathbf{i}'\mathbf{w} &= 1 \text{ is the tangency portfolio.}
\end{aligned}$$

As we can see from the last equality the portfolio frontier is a straight line again.

In equilibrium, the demand for risky assets (weights in the tangency portfolio) should be equal to the supply which is equal to the value of the offered shares. (the number of outstanding shares $\times$ price).

To sum up recall that each agent solves the following problem

$$\begin{aligned}
& \max_z \ \mu_p - \alpha_i.\sigma_p^2 \\
\textit{where } \mu_p &= z.\mu_T + (1-z).r^0 \\
\sigma_p^2 &= z^2.\sigma_T^2
\end{aligned}$$

In equilibrium, tangency portfolio is the market portfolio (usually S&P500)

Example T(50%,50%)

A:100	60%	40%	(20%,20%)
B:100	50%	50%	(25%,25%)

One such model is CAPM :

$$\begin{aligned}\mu_i - r^0 &= \beta_i(\mu_m - r^0)\\ \beta_i &= \frac{cov(r_i, r_m)}{var(r_m)}\end{aligned}$$

Recall from the previous chapter, if we add the time dimension to the general model; simplest case today and tomorrow the financial problem becomes

$$\max_{y^0, y} \; u(y^0) + \delta E(u(y))$$

$$\begin{aligned}0 &\geq p_0^{'}(x^0 - w^0) + q.z\\ 0 &\geq p_s^{'}(x^s - w^s) + r_s.z) \quad \text{s=1,2...S}\end{aligned}$$

If the asset market clears, we can apply the reverse decomposition and express the problem i.t.o. Arrow securities.

$$\max_{y^0, y} \; u(y^0) + \delta E(u(y))$$

$$0 \geq y^0 - w^0 + \sum_s \pi_s.\alpha_s(y^s - w^s)$$

HARA (Hyperbolic Absolute Risk Aversion)

A quite common utility function is HARA. It is the general form of the utility and with appropriate selection of the parameters we can get other types log, power, etc.

Definition $T(w) = \frac{1}{A(w)}$ **absolute risk tolarance**.

Proposition A utility is HARA type if $T^{'}(w) = a + bw$. Note that it is linear to the wealth.

If we derive the FOC of the above financial problem

$$\begin{aligned}\frac{\partial L}{\partial y^0} &: \; u^{'}(y^0) = \lambda\\ \frac{\partial L}{\partial y^s} &: \; \pi_s \delta u^{'}(y^s) = \lambda \alpha_s \quad \text{s=1,2...S}\end{aligned}$$

If we take a HARA utility function

$$\begin{aligned}u(x) &= \frac{1}{b-1}(a + bx)^{\frac{b-1}{b}}\\ u^{'}(x) &= (a + bx)^{-\frac{1}{b}}\end{aligned}$$

Applying to the financial problem

$$\begin{aligned}(a+by^0)^{-\frac{1}{b}} &= \lambda \\ \pi_s.\delta(a+by^s)^{-\frac{1}{b}} &= \lambda.\alpha_s\end{aligned}$$

$$\begin{aligned}\sum_s^S \alpha_s.y^s &= \text{"}saving\text{"} = \theta_0 + \theta_1 \bar{w} \\ \bar{w} &= w^0 + \sum_s^S \alpha_s.w^s\end{aligned}$$

Note that the coefficients θ_0, θ_1 does not depend on the initial wealth. So in case of HARA utility, saving is linear in wealth.

We followed the mean-variance framework in portfolio choice instead of the Arrow-Debreu framework, since the former allowed us to focus on the real assets in the portfolio instead of the artificial Arrow securities.