Lecture 1 / Week 1

Classes of Events

Definition A **random experiment** is an experiment where the outcome cannot be predicted in advance.
Example We can observe the price of an asset at t_0, price in [0,T]

Definition The set of all the possible outcomes is called **sample space** Ω.
Example The price at time t_0 : Ω = [0,∞]
Log return : $\Omega = (-\infty,\infty)=\mathbb{R}$ (the set of all real numbers),
Note:$[-\infty,\infty]$ = extended real line

Example Observe a price between time 0 and time T. If the price moves continuously $\Longrightarrow \Omega = \{w = w(t) \;\; w : [0, T] \to [0, \infty],\ w \text{ is continuous}\}$

Insert here Figure 1

Definition Let Ω be a sample space. An **event** is a subset of Ω. In other words it is the set of possible outcomes of the experiment.
Example Price at t_0 ; Event A=[0,a) . So the event says, the price at t_0 is lower than a. Price in [0,T].
An event could also be A: the price at $\frac{T}{2}$ is lower than a. Formally, A= $\{w = w(t) : w(\frac{T}{2}) < a\}$

Insert here Figure 2

Definition $\cup_{i\in I}A_i$ = The union of A_i's (at least one of the A_i's)
Definition $\cap_{i\in I}A_i$ = The intersection of A_i's (all of the A_i's)
Definition A^C = The complement of A(not A).
Example In the previous example, A^C = the price at $\frac{T}{2} \geq a$.Formally, A^C= $\{w = w(t) : w(\frac{T}{2}) \geq a\}$

Insert here Figure 3

Definition A **class** of events is a set of events with certain properties.
Definition Let Ω be the sample space and $\mathcal{F}$ a class of events in Ω. $\mathcal{F}$ is a **sigma-algebra** (**σ-algebra**) if and only if it has the following properties:

1. $\Omega \in \mathcal{F}$. (So in words, the entire set should belong to the class)
2. If A$\in \mathcal{F}$, then $A^C \in \mathcal{F}$.(If an event belongs to the class, then its complement should also belong to the class.)
3. If $A_1, A_2, ... \in \mathcal{F}$,then $\cup_{i=1}^{\infty} A_i \in \mathcal{F}$.(If a sequence of events belong to the class, then their countable union should also belong to the class.(Note: $\cup^{\infty}$:countable union, $\cup^n where\ n \in \mathbb{N}$: finite union)

Insert here Figure 4

Remark $\mathcal{F}$ is closed under complements and unions.

Claim A σ-algebra is closed under intersections.

Proof Suppose we have a sequence of events $A_1, A_2, ... \in \mathcal{F}$, then (by property 2) $A_1^C, A_2^C, ... \in \mathcal{F}$.Then, by property 3, we know that $\cup_{i=1}^{\infty} A_i^c \in \mathcal{F}$. Using the De Morgan's Law, $\cup_{i=1}^{\infty} A_i^C = (\cap_{i=1}^{\infty} A_i)^C \in \mathcal{F}$.Let's call $(\cap_{i=1}^{\infty} A_i)^C =$ $B \in \mathcal{F}$, but then $B^C = \cap_{i=1}^{\infty} A_i \in \mathcal{F}$. *QED*.

Claim $\mathcal{F}$ is closed under unions iff(if and only if) $\mathcal{F}$ is closed under intersections. (Given property 1 and property 2)

Proof '$\Rightarrow$' : This implication we have just proved.

'$\Leftarrow$': Suppose we have a sequence of events $A_1, A_2, ... \in \mathcal{F}$,then (by property 2) $A_1^C, A_2^C, ... \in \mathcal{F}$.By hypothesis, we know $\cap_{i=1}^{\infty} A_i^c \in \mathcal{F}$. Using the De Morgan's Law, $\cap_{i=1}^{\infty} A_i^C = (\cup_{i=1}^{\infty} A_i)^C \in \mathcal{F}$. Then by property 2, $(\cup_{i=1}^{\infty} A_i) \in \mathcal{F}$. *QED*.

Remark We know that $\mathcal{F}$ is closed under countable unions(intersections). Can we also say that it is closed under finite unions(intersections)? $A_1, A_2, ...A_n \in \mathcal{F} \Rightarrow (?) \cup_{i=1}^{N} A_i \in \mathcal{F}$. Yes, indeed we can!

Proof By property 1, we know that $\Omega \in \mathcal{F}$. Since $\Omega^C = \emptyset$, but then $\emptyset \in \mathcal{F}$. So, we can take the infinite sequence $A_1, A_2, ...A_n, \emptyset, \emptyset, \emptyset......$ Then, $\cup_{i=1}^{\infty} A_i \in \mathcal{F}$ implies $\cup_{i=1}^{n} A_i \in \mathcal{F}$. *QED*.

Definition A class of events that is closed under complements and finite unions denoted as $\mathcal{A}$ where $\Omega \in \mathcal{A}$ is called an **algebra (field)** of events.

Remark If a class of events is a σ-algebra, then it is an algebra, but reverse is not generally true. Formally, σ-$ALG \Rightarrow ALG$, but $ALG \nRightarrow \sigma$-ALG. If Ω is finite, an algebra is is also a σ-algebra.

Example $\Omega = (0, 1]$

$\mathcal{A} = \{$finite unions of intervals of the type (a,b] with $0 \leq a \leq b \leq 1\}$

Exercise $\mathcal{A}$ is an algebra.

Proof We have to check whether it satisfies the 3 properties of algebra. Property 1: Taking a=0 and b=1, we show that $\Omega \in \mathcal{A}$. Property 3: $\cup_{i=1}^{n} (a_i, b_i] \in \mathcal{F}$. By induction, we can see that the finite union belongs to the algebra. When n=1, it holds by definition. If we take $(a_1, b_1] \cup (a_2, b_2]$. We get a union of the same form. If this holds for n and it can be shown that it also holds for n+1. Property 2: The complements of the sets (a,b], have the form $(a,b]^C = (0, a] \cup (b, 1]$. Since they are union of elements of $\mathcal{A}$, they belong to the algebra.

Remark $\mathcal{A}$ is not a σ-algebra.

Proof Take sets of the form: $\cap_2^\infty(\frac{1}{2}-\frac{1}{n},1] \Rightarrow (0,1]\cap(\frac{1}{2}-\frac{1}{3},1]\cap .... = [\frac{1}{2},1] \notin \mathcal{A}$,(={x$\in \mathbb{R}: \frac{1}{2}-\frac{1}{n} < x \leq 1$ for every n}), since the countable intersection does not belong to $\mathcal{A}$, the class is not a σ-algebra. QED.

Definition Let Ω be a sample space. Suppose $\mathcal{C}$ is a class of events.

$$\sigma(\mathcal{C}) = \cap_{\mathcal{G} \text{ is } \sigma\text{-}ALG \text{ and } \mathcal{G} \supset \mathcal{C}} \mathcal{G},\ \ \sigma(\mathcal{C}) \text{ is called } \sigma\text{-algebra } \textbf{generated by } C.$$

1) $\sigma(\mathcal{C})$ is a σ-algebra.

2) It is the smallest σ-algebra containing $\mathcal{C}$.

Proof Property (1): Every intersection of σ-algebra is a σ-algebra. Suppose $\{\mathcal{F}_\theta\}_{\theta\in\Theta}$, $\cap_{\theta\in\Theta}\mathcal{F}_\theta = \mathcal{F}$.

1) $\Omega \in \mathcal{F}_\theta\ \forall\ \theta \Rightarrow \Omega \in \cap_{\theta\in\Theta}\mathcal{F}_\theta$

2)$A \in \mathcal{F} = \cap_{\theta\in\Theta}\mathcal{F}_\theta \Rightarrow A \in \mathcal{F}_\theta\ \forall\ \theta \Rightarrow A^C \in \mathcal{F}_\theta\ \ \forall\ \theta \Rightarrow A^C \in \cap_{\theta\in\Theta}\mathcal{F}_\theta = \mathcal{F}$

3) If $A_1, A_2, ... \in \mathcal{F} = \cap_{\theta\in\Theta}\mathcal{F}_\theta \Rightarrow A_1, A_2, ... \in \mathcal{F}_\theta\ \forall\ \theta \Rightarrow \cup_{i=1}^\infty A_i \in \mathcal{F}_\theta\ \forall\ \theta \Rightarrow \cup_{i=1}^\infty A_i \in \cap_{\theta\in\Theta}\mathcal{F}_\theta = \mathcal{F}$. QED.

Property (2): If $\mathcal{F}$ is σ-ALG $\supseteq \mathcal{C} \Rightarrow \mathcal{F} \supseteq \sigma(\mathcal{C}) \Rightarrow \mathcal{F}$ is in the intersection, so $\sigma(\mathcal{C})$ must be the smallest σ-ALG. QED

Definition Let $\Omega = \mathbb{R}$, $\mathcal{C} = \{(a,b) : -\infty < a < b < \infty\}$ a general class(Note that it is not σ-algebra). The **Borel** σ-algebra on $\mathbb{R} = \sigma(\mathcal{C}) \Rightarrow B(\mathbb{R})$.The **Borel σ-algebra** on $\mathbb{R}$ is the σ-algebra generated by C. Note that it contains the singletons {a}.

$$\begin{aligned}
\{a\} &= \cap_{n=1}^\infty(a-\frac{1}{n}, a+\frac{1}{n}) \in B(\mathbb{R}).\\
(a,b] &= (a,b)\cup\{b\} \in B(\mathbb{R}).smilarly, [a,b), [a,b] \in B(\mathbb{R})\\
(a,b] &\in B(\mathbb{R}), [a,b] \in B(\mathbb{R}),\\
All\ finite\ unions\ of\ intervals &\in B(\mathbb{R})\\
(-\infty, a] &= \cup_{n=1}^\infty (a-n, a] \in B(\mathbb{R})\\
\mathcal{C}^{'} &= \{(a,b] : -\infty < a < b < \infty\}\ \ \sigma(\mathcal{C}^{'}) = B(\mathbb{R})\\
\mathcal{C}^{"} &= \{(\infty, a] : -\infty < a < \infty\}\ \ \ \ \sigma(\mathcal{C}^{"}) = B(\mathbb{R})\ \ \text{(Proof p.14)}
\end{aligned}$$

$\mathbb{R}^K = \{(x_1, x_2, ..., x_K) : x_i \in \mathbb{R}\}$

$\mathcal{C} = \{(a_1,b_1)\text{x}(a_2,b_2)\text{x}.....(a_K,b_K) : -\infty < a_i < b_i < \infty\}$

$(x_1, x_2, ..., x_K) : x_i \in (a_i, b_i)\ \ \forall\ i.$

Insert figure here 5

Remark $B(\mathbb{R}^K) = \sigma(\mathcal{C})$ contains singletons, finite and countable sets,open sets,closed sets...

Definition Let Ω be a sample space. Fix a σ-algebra $\mathcal{F}$. $\mathcal{F}$ contains all the relevant events $A_1, A_2, ..., \in \mathcal{F}, \cup_{i=1}^{\infty} A_i \in \mathcal{F}$.

A **probability measure** is a function P:$\mathcal{F} \to \mathbb{R}$ that satisfies

1) P(A)≥ 0

2) P(Ω)=1

3) If $A_1, A_2,$....are disjoint events then $P(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$ where $A_i \cap A_j = \emptyset$, i$\neq$j. (σ-additive)

Example Observe a price from time 0 to T.(Note that the sample space Ω is a set of functions).$\mathcal{F}$ is a σ-algebra of all events.

$\mathcal{C} = \{$price at time t is lower than a (p_t <a), $\forall$ a$\in \mathbb{R}$,$\forall$ t$\in [0, T]\}$, $\mathcal{F} = \sigma(\mathcal{C})$.

Insert here Figure 6

Definition Events until time time t_o : $\sigma\{$price at time t is smaller than a,$\forall$ a$\in \mathbb{R}$,$\forall$ t $\leq t_o\} = \mathcal{F}_{t_0}$. Note that $\mathcal{F}_{t_0} \subseteq \mathcal{F}$. $\mathcal{F}$ is the **universe σ-algebra** and $\mathcal{F}_t$ is the **sub σ-algebra** of $\mathcal{F}$.

Definition $\mathcal{F}_0 = \{\emptyset, \Omega\}$ is the smallest σ-algebra of subsets of Ω.

Definition $\mathcal{P}(\Omega) = \{$All the subsets of $\Omega\}$ is the biggest σ-algebra.(**Part σ-algebra**).

When Ω =$\mathbb{R}$, we can choose $\mathcal{P}(\mathbb{R})$, but in general $B(\mathbb{R})$ is chosen instead, since it is smaller. It can be proved that $B(\mathbb{R}) \subseteq \mathcal{P}(\mathbb{R})$. If we choose $\mathcal{P}(\mathbb{R})$, then we cannot define a sensible probability measure on it. The only probability on $\mathcal{P}(\mathbb{R})$ is the discrete probability, where only the points have positive probability.

Insert here Figure 7

Remark We have already shown that $\cap_{\theta \in \Theta} \mathcal{F}_\theta$ is a σ-algebra. Yet, $\cup_{\theta \in \Theta} \mathcal{F}_\theta$ is not a σ-algebra. But still we can obtain a σ-algebra generated by $\cup_{\theta \in \Theta} \mathcal{F}_\theta$. $\Rightarrow \sigma(\cup_{\theta \in \Theta} \mathcal{F}_\theta) = \vee_{\theta \in \Theta} \mathcal{F}_\theta$. This σ-algebra is generated in the same way we have seen before(Definition 10); take a class $\mathcal{C}$, then $\sigma(\mathcal{C}) = \cap_{\mathcal{G} \text{ is } \sigma-ALG \text{ and } \mathcal{G} \supset \mathcal{C}}$ $\mathcal{G}$, we only have to set $\mathcal{C} = \cup_{\theta \in \Theta} \mathcal{F}_\theta$.

Example The following example helps us to see that $\cup_{\theta \in \Theta} \mathcal{F}_\theta$ is not a σ-algebra.

Take the following σ-algebra: $\mathcal{F}_\theta = \{\emptyset, \mathbb{R}, (-\infty, \theta], (\theta, \infty)\}$, then $\cup_{\theta \in \Theta} \mathcal{F}_\theta = \{\emptyset, \mathbb{R}, (-\infty, \theta], (\theta, \infty) \quad \theta \in \mathbb{R}\}$

If we take $\cup_{n=1}^{\infty}(-\infty, -\frac{1}{n}] = (-\infty, 0) \notin \cup_{\theta \in \Theta} \mathcal{F}_\theta$, even though $(-\infty, -\frac{1}{n}] \in \cup_{\theta \in \Theta} \mathcal{F}_\theta$

Remark If we have two classes such that $\mathcal{C}_1 = \{(0, 1), (2, 3), (5, 6)\}$ and $\mathcal{C}_2 = \{(-1, 0), (7, 8)\}$, then $\mathcal{C}_1 \cup \mathcal{C}_2 = \{(0, 1), (2, 3), (5, 6), (-1, 0), (7, 8)\}$.

Insert here Figure 8