Lecture 2 / Week 1

Random Variables

Definition (General Definition of Measurability): Suppose we have Ω, $\Omega^{'}$ sample spaces and $\mathcal{F}_{\Omega}$, $\mathcal{F}^{'}_{\Omega'}$ σ-algebras. Then we define the following mapping:

$$g:\ \Omega \rightarrow \Omega^{'}$$

Insert here Figure 1

Inverse Image of B (a subset of $\Omega^{'}$) through g: The set $\{\ \omega \in \Omega : g(\omega) \in B\ \} = g^{-1}(B)$. Note that the inverse image is not necessarily a inverse function. It is just the inverse of the direct image.

Definition g is $\mathcal{F}\backslash\mathcal{F}^{'}$**measurable** if for every $B \in \mathcal{F}^{'}$, $g^{-1}(B) \in \mathcal{F}$.
Remark An interesting case is if $\Omega^{'} = \mathbb{R}$.

Definition Suppose we have a sample space Ω, σ-algebra $\mathcal{F}$, and probability measure P, then $(\Omega, \mathcal{F}, P)$ is a **probability space**.

Definition A **random variable** is a function

$$\mathcal{X}:\ \Omega \rightarrow \mathbb{R}$$
which is $\mathcal{F} \setminus \mathcal{B}(\mathbb{R})$ measurable.

We want to be able to define the probability of $P(\mathcal{X} \in B)$, i.e the probability of events like $(\mathcal{X} \in B)=\{\ \omega \in \Omega : \mathcal{X}(\omega) \in B\ \}= \mathcal{X}^{-1}(B)$. Note that B is a Borel set of the Borel σ-algebra $\mathcal{B}(\mathbb{R})$. For example, such a set could be B=(0,1) and then $(\mathcal{X} \in (0,1))= (0<\mathcal{X} <1)$. Also note that $(\mathcal{X} \in B)=\mathcal{X}^{-1}(B) \in \mathcal{F}$.

Insert here Figure 2

Definition Let $(\Omega, \mathcal{F}, P)$ be a probaility space and $\mathcal{G} \subseteq \mathcal{F}$, where $\mathcal{G}$ is a sub σ-algebra of $\mathcal{F}$. Then $\mathcal{X}$ is $\mathcal{G}$**-measurable** if for every $B \in \mathcal{B}(\mathbb{R})$,

$$\mathcal{X}^{-1}(B) \in \mathcal{G}$$
$$(\mathcal{X} \in B) \in \mathcal{G}$$

Theorem Suppose we a sample space Ω and a σ-algebra $\mathcal{F}$, of subsets of Ω. Let $\mathcal{X}$ be a function s.t $\mathcal{X}:\ \Omega \rightarrow \mathbb{R}$. Then $\mathcal{X}$ is $\mathcal{F} \setminus \mathcal{B}(\mathbb{R})$ measurable if and only if $\forall\ y \in \mathbb{R}$,

$$A_y = \{\omega \in \Omega : \mathcal{X}(\omega) \leq y\} \in \mathcal{F}$$

In general, we can try to take every Borel set B, compute $\mathcal{X}^{-1}$(B) and check if $\mathcal{X}^{-1}$(B) $\in \mathcal{F}$. But this would be very cumbersome, instead the theorem tells us that we can just take the inverse images of particular class of sets such as $\mathcal{X}^{-1}(-\infty, y] = \{\omega \in \Omega : \mathcal{X}(\omega) \leq y\} = \{\omega : \mathcal{X}(\omega) \in (-\infty, y]$ and check whether they belong to $\mathcal{F}$.

Proof Take the class D={B $\in \mathcal{B}(\mathbb{R}) : \mathcal{X}^{-1}$(B) $\in \mathcal{F}$}. Note that it is a set of Borel sets s.t the inverse image belongs to the σ-algebra. It is easy to verify that D is a σ-algebra. (Exercise!) Notice that (-∞, y]$\in$ D and {(-∞, y]: $y \in \mathbb{R}$} $\subseteq$ D. We know that $\mathcal{B}(\mathbb{R})$ is the smallest σ-algebra containing these intervals. By definition, we can generate $\mathcal{B}(\mathbb{R})$ by using intervals (-∞, y].So, $\mathcal{B}(\mathbb{R}) \subseteq$ D. Since by construction of D, D $\subseteq \mathcal{B}(\mathbb{R}) \Rightarrow \mathcal{B}(\mathbb{R})$ = D. But then, $\forall$ B $\in \mathcal{B}(\mathbb{R})$, B $\in$ D and by construction $\mathcal{X}^{-1}$(B) $\in \mathcal{F} \Rightarrow$ (*By definition measurability*) $\mathcal{X}$ is $\mathcal{F} \setminus \mathcal{B}(\mathbb{R})$ measurable.

Exercise Verify that D is a σ-algebra.

Proof To verify that D is a σ-algebra, we have to check whether it satisfies 3 properties of σ-algebra: 1.) Choose $\mathbb{R} \in \mathcal{B}(\mathbb{R})$, then $\mathcal{X}^{-1}(\mathbb{R}) = \Omega \in \mathcal{F}$, since $\mathcal{F}$ is a σ-algebra. 2.) Pick any B, if B $\in$D, then B^C should also belong to D. Since $\mathcal{X}^{-1}$(B^C) = ($\mathcal{X}^{-1}$(B))^C and $\mathcal{F}$ is a σ-algebra, then B$^C \in D$.3.)if B$_1, B_2$.... $\in$D, then it should hold that $\cup_{i=1}^{\infty} B_i \in D$. Since $\mathcal{X}^{-1}(\cup_{i=1}^{\infty}\text{B}_i) = \cup_{i=1}^{\infty}\mathcal{X}^{-1}(\text{B}_i) \in \mathcal{F}$, $\cup_{i=1}^{\infty} B_i \in D$.QED.

Theorem If $\mathcal{X}_1, \mathcal{X}_2$.... are $\mathcal{F} \setminus \mathcal{B}(\mathbb{R})$ measurable, then

1. min($\mathcal{X}_1$,$\mathcal{X}_2$,....,$\mathcal{X}_n$), max ($\mathcal{X}_1$,$\mathcal{X}_2$,....,$\mathcal{X}_n$) are measurable.
2. $\sup_n \mathcal{X}_n$, $\inf_n \mathcal{X}_n$ are measurable.
3. lim $\inf_n \mathcal{X}_n$, lim $\sup_n \mathcal{X}_n$ are measurable.
4. $\lim_{n\to\infty}\mathcal{X}_n$ if it exists is measurable.

Proof (1) min($\mathcal{X}_1, \mathcal{X}_2, ..., \mathcal{X}_n$) is $\mathcal{F} \setminus \mathcal{B}(\mathbb{R})$ measurable. $\forall$ $y \in \mathbb{R}$, using the previous Theorem we know that it is enough to show that {$\omega \in \Omega$: min($\mathcal{X}_1(\omega)$,$\mathcal{X}_2(\omega)$, ...,$\mathcal{X}_n(\omega)$) $\leq y$ } $\in \mathcal{F}$. Note that, min($\mathcal{X}_1$,$\mathcal{X}_2$, ...,$\mathcal{X}_n$) $\leq y$) = $\cup_{j=1}^{n}$($\mathcal{X}_j$ $\leq y$). This means that ($\mathcal{X}_j \leq y$) for at least one j. By hypothesis ($\mathcal{X}_j \leq y$) $\in \mathcal{F}$ for $\forall$ $j \Rightarrow \cup_{j=1}^{n}(\mathcal{X}_j \leq y) \in \mathcal{F}$. The max($\mathcal{X}$ $_1$, $\mathcal{X}$ $_2$, ..., $\mathcal{X}$ $_n$) can be proved the same way; i.e {$\omega \in \Omega$: max($\mathcal{X}_1(\omega)$,$\mathcal{X}_2(\omega)$, ...,$\mathcal{X}_n(\omega)$) $\leq y$ } $\in \mathcal{F}$. Note that, max($\mathcal{X}_1$,$\mathcal{X}_2$, ...,$\mathcal{X}_n$) $\leq y$) = $\cap_{j=1}^{n}(\mathcal{X}_j \leq y)$. This means that ($\mathcal{X}_j \leq y$) for all j. By hypothesis ($\mathcal{X}_j \leq y$) $\in \mathcal{F}$ for $\forall$ $j \Rightarrow \cap_{j=1}^{n}(\mathcal{X}_j \leq y) \in \mathcal{F}$.

(2) since ($\sup_n \mathcal{X}_n \leq y$) = $\cap_{n=1}^{\infty}$ ($\mathcal{X}_n \leq y$). This means that ($\mathcal{X}_n \leq y$) for all n. By hypothesis ($\mathcal{X}_n \leq y$) $\in \mathcal{F}$ for $\forall$ $n \Rightarrow \cap_{n=1}^{\infty}(\mathcal{X}_n \leq y) \in \mathcal{F}$, similarly since ($\inf_n \mathcal{X}_n \leq y$) = $\cup_{n=1}^{\infty}$ ($\mathcal{X}_n \leq y$). This means that ($\mathcal{X}_n \leq y$) for all n. By hypothesis ($\mathcal{X}_n \leq y$) $\in \mathcal{F}$ for $\forall$ $n \Rightarrow \cup_{n=1}^{\infty}(\mathcal{X}_n \leq y) \in \mathcal{F}$

(3) $\lim_{n\to\infty} \inf_{m\geq n}\mathcal{X}_m$. =$\sup_{n\in\mathbb{N}}(\inf_{n\geq m} \mathcal{X}_m)$ and $\lim\sup_n \mathcal{X}_n = \inf_{n\in\mathbb{N}}(\sup_{n\geq m} \mathcal{X}_m)$ and follows from the previous proof.

(4) a$_n$ $\lim_{n\to\infty} a_n$ does not always exist. It exists iff lim $\sup_n$ a_n =lim $\inf_n$ a_n, then follows from previous proof.

Definition A **simple random variable** is a function that takes finite number of values. (Suppose $A_i \cap A_j = \emptyset$, $i \neq j$)

$$\mathcal{X}(\omega) = \left\{ \begin{array}{ll} a_1 & \omega \in A_1 \\ a_2 & \omega \in A_2 \\ & . \\ & . \\ a_n & \omega \in A_n \end{array} \right\}$$

Definition The **indicator function** is defined as follows: ($A \in \mathcal{F}$)

$$1_A(\omega) = \left\{ \begin{array}{l} 1 \ \text{ if } \omega \in A \\ 0 \ \text{ if } \omega \notin A \end{array} \right\}$$

$$\boxed{\mathcal{X}(\omega) = \textstyle\sum_{i=1}^{n} a_i 1_{A_i}(\omega)}$$

Example $\mathcal{X}(\omega) = \left\{ \begin{array}{ll} 1 & \omega \in A_1 \\ 2 & \omega \in A_2 \\ 3 & \omega \in A_2 \end{array} \right\}$

$$\mathcal{X}(\omega) = 1 * 1_{A_1} + 2 * 1_{A_2} + 3 * 1_{A_3}$$

Remark A simple random variable is $\mathcal{F} \setminus \mathcal{B}(\mathbb{R})$ measurable. (Verify!)

Theorem A function $\mathcal{X}$: $\Omega \to \mathbb{R}$ is $\mathcal{F} \setminus \mathcal{B}(\mathbb{R})$ measurable if and only if there exists a sequence $(\mathcal{X}_n)$ of simple random variables such that for every $\omega \in \Omega$,

$$\mathcal{X}(\omega) = \lim_{n\to\infty} \mathcal{X}_n(\omega)$$

Proof " $\Leftarrow$ " : Let $\mathcal{X}(\omega) = \lim_{n\to\infty} \mathcal{X}_n(\omega)$ and $\mathcal{X}_n$ be a simple random variable. Previous remark tells us $\mathcal{X}_n$ is measurable and by point 4 of the previous theorem we know that $\lim_{n\to\infty} \mathcal{X}_n(\omega)$ is measurable, so $\mathcal{X}(\omega)$ is measurable. *QED*.

" $\Rightarrow$ " : We want to prove that if $\mathcal{X}(\omega)$ is $\mathcal{F} \setminus \mathcal{B}(\mathbb{R})$ measurable, then $\mathcal{X}(\omega) = \lim_{n\to\infty} \mathcal{X}_n(\omega)$ is measurable, where $\mathcal{X}_n$is simple random variable. First we suppose $\mathcal{X} \geq 0, \forall\, \omega\ \mathcal{X}(\omega) \geq 0$. We prove the statement under this condition and then show that it also holds in the opposite case where $\mathcal{X}(\omega) \leq 0$.

Take $\mathcal{X}_n(\omega) = \left\{ \begin{array}{ll} \frac{k}{2^n} & \text{if } \frac{k}{2^n} \leq \mathcal{X}(\omega) < \frac{k+1}{2^n} \\ 0 & \text{otherwise} \end{array} \right\}$

k=0,1,2,.......$n2^n - 1$

Fix n : $\mathcal{X}_n$ takes values $0, \frac{1}{2^n}, \frac{2}{2^n} \ldots . n$

$\mathcal{X}_n$ is simple when $n \to \infty$, $\mathcal{X}_n(\omega) \to \mathcal{X}(\omega)$. (See the figure below for $\mathcal{X}_1(\omega)$ and $\mathcal{X}_2(\omega)$) This completes the proof when $\mathcal{X}(\omega) \geq 0$.

Insert here Figure 3

To see that the result still holds in the opposite case $\mathcal{X}(\omega) \leq 0$; one should see that the function $\mathcal{X} = \mathcal{X}^+ - \mathcal{X}^-$, where $\mathcal{X}^+(\omega) = \max(\mathcal{X},0) \geq 0$ and $\mathcal{X}^-(\omega) =$

$-\min(\mathcal{X},0) \geq 0$ (This can be best seen sketching the graph of those functions.). But, then $\mathcal{X}^+ = \lim_n \mathcal{X}_n^+$, $\mathcal{X}^- = \lim_n \mathcal{X}_n^-$ and $\mathcal{X} = \mathcal{X}^+ + \mathcal{X}^- = \lim_n(\mathcal{X}_n^+ - \mathcal{X}_n^-)$ completes the proof. *QED*.

Corollary If $\mathcal{X}_1, \mathcal{X}_2$ are $\mathcal{F} \setminus \mathcal{B}(\mathbb{R})$ measurable, then $\mathcal{X}_1 + \mathcal{X}_2, \mathcal{X}_1 - \mathcal{X}_2, \mathcal{X}_1 * \mathcal{X}_2, \mathcal{X}_1/\mathcal{X}_2$(Provided $\mathcal{X}_2 \neq 0$) are $\mathcal{F} \setminus \mathcal{B}(\mathbb{R})$ measurable.

Proof $(\mathcal{X}_1 + \mathcal{X}_2)$: Enough to observe that $\mathcal{X}_1 = \lim_{n_1 \to \infty} \mathcal{X}_{n_1}$ and $\mathcal{X}_2 = \lim_{n_2 \to \infty} \mathcal{X}_{n_2}$, because then $\mathcal{X}_1 + \mathcal{X}_2 = \lim \mathcal{X}_{n_1} + \lim \mathcal{X}_{n_2} = \lim\ (\mathcal{X}_{n_1} + \mathcal{X}_{n_2})$. This follows from limit property that sum of two limits equals to the limit of the sum. Moreover, the fact that the sum of simple functions is a simple function and the above theorem guarantees that $(\mathcal{X}_1 + \mathcal{X}_2)$ is $\mathcal{F} \setminus \mathcal{B}(\mathbb{R})$ measurable.