Lecture 6 / Week 4

OUTLINE

1) Examples of Conditional Expectation.
2) Definition of of Conditional Expectation. *Book* 3.1
3) Properties of Conditional Expectation *Book* §3.2

Conditional Expectation

Example 1 Consider the following Game of Chance: A coin is tossed 10 times and the winnings of the game is denoted by $Y = 1\$$ per head. There are two cases: 1.) The player can enter the game soon. 2.) The player can enter the game after the first 6 tosses. (still receives the gain of previous tosses.)

Question: What is the fair price of the game in both cases?

Answer Case 1: The fair price of the game= 5\$=$E(Y)$: expected winnings of the game.

Case 2: Define X as the number of heads in the first 6 tosses, then the fair price would be $E(Y \mid X) = X+2$: the expected winnings given the information about the outcome of the first 6 tosses, e.g. if there were 6 heads in the first 6 tosses than $E(Y \mid X) = 8$.

Example 2 (Information is given by a σ-algebra, not a random variable). Consider the following Game of Chance: A box contains a couple of dice: read dice and blue dice. The blue dice is numbered from 1-6 and the red dice is numbered from 7-12. We take a dice randomly and throw it. The winnings of the game is denoted by $Y =$ the score of the dice in \$. There are two cases: 1.) The player can enter the game at the beginning. 2.) The player can enter the game after having seen the color of the dice.

Question: What is the fair price of the game?

Answer Case 1: At the beginning of the game:$E(Y) : \frac{1+2+3+4+5+6}{12} = 6.5\$$

Case 2: After having seen the color of the dice:

$$\begin{aligned} E(Y|\mathcal{F}_0) \text{ where } \mathcal{F}_0 &= \{\emptyset, \Omega, Blue, \mathrm{Re}\,d\} \\ E(Y|\mathcal{F}_0) &= \left\{ \begin{array}{c} Blue\text{: } \frac{1+2+3+4+5+6}{6} = 3.5 \\ \mathrm{Re}\,d : \frac{7+8+9+10+11+12}{6} = 9.5 \end{array} \right\} \end{aligned}$$

Observations

- $E(Y|\mathcal{F}_0)$ is a random variable.

- Its value is completely determined if we know which events of $\mathcal{F}_0$ occur, i.e. once we know the color in the above example we can calculate the conditional expectation.
- $E(Y|\mathcal{F}_0)$ has the meaning of expectation.

Definition of Conditional Expectation

Definition Suppose that Y is an *integrable* random variable defined on the probability space $(\Omega, \mathcal{F}, P)$. Let $\mathcal{F}_0$ be a sub σ-algebra of $\mathcal{F}$. Then, the **conditional expectation** of Y given $\mathcal{F}_0$ is a random variable Z on same probability space $(\Omega, \mathcal{F}, P)$ such that

1. Z is $\mathcal{F}_0-$ measurable.
2. For every event $A \in \mathcal{F}_0 \rightarrow \int_A Z dP = \int_A Y dP$

The first property can be interpreted using the previous example, once i know the color (an event in $\mathcal{F}_0$), i can find which value Z takes,i.e. $Z = z$. The second property just says that the mean value of Y and Z is the same.

Definition If we know completely about the experiment, (Blue dice and 5), then we have complete information, formally

$$\textbf{Full information} := E(Y|\mathcal{F})$$

while we have partial information if we only know the color of the dice, but not the number itself, formally

$$\textbf{Partial Information} := E(Y|\mathcal{F}_0)$$

Example We have the following sample space $\Omega = \{1, 2, 3, 4, 5, 6, \mathbf{7}, \mathbf{8}, \mathbf{9}, \mathbf{10}, \mathbf{11}, \mathbf{12}\}$. The numbers 1-6 are in the blue dice, 7-12 in the red dice. We can define two different random variables.

$$\begin{aligned} Random\ Variable 1 &: \quad Y = 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11\ 12 \\ Random\ Variable 2 &: \quad E(Y|\mathcal{F}_0) = 3,5\ 3,5\ 3,5\ 3,5\ 3,5\ 3,5 \mid \mathbf{9.5}\ \mathbf{9.5}\ \mathbf{9.5}\ \mathbf{9.5}\ \mathbf{9.5}\ \mathbf{9.5} \end{aligned}$$

The first random variable can takes values from 1 to 12, whereas the second one (which is the conditional expectation, but a random variable itself,let's say

Z) takes either the value 3.5 or 9.5. Then the mean values depending on the event (blue or red)

$$\begin{array}{rll} \text{On Blue} & : \text{mean value of} & E(Y|\mathcal{F}_0) = 3.5 \\ \text{On Blue} & : \text{mean value of} & Y = 3.5 \\ \text{On Red} & : \text{mean value of} & E(Y|\mathcal{F}_0) = 9.5 \\ \text{On Red} & : \text{mean value of} & Y = 3.5 \end{array}$$

This explains the second property of the definition that the mean values are the same w.r.t the same event in the σ-algebra.

Notation Almost surely **(a.s) =** with probability 1

Examples

$$\begin{array}{rcl} Z_1 & = & Z_2 \text{ a.s } \Leftrightarrow P\{\omega : Z_1(\omega) = Z_2(\omega)\} = 1 \\ Z_1 & \leq & Z_2 \text{ a.s } \Leftrightarrow P\{\omega : Z_1(\omega) \leq Z_2(\omega)\} = 1 \\ Z_{n_{n\to\infty}} & \to & Z \text{ a.s } \Leftrightarrow P\{\omega : Z_{n_{n\to\infty}}(\omega) \to Z(\omega)\} = 1 \end{array}$$

Proposition $E(Y|\mathcal{F}_0)$ exists whenever Y is integrable.

Proof Hint: Take a random variable Y and a σ-algebra $\mathcal{F}_0$ it is possible to find a Z that satisfies the properties of the definition of conditional expectation, i.e Z is $\mathcal{F}_0-$ measurable. and every event $A \in \mathcal{F}_0 \to \int_A Z dP = \int_A Y dP$.

The question is, is Z uniquely determined?

The answer is almost, we can find Z_1 and Z_2 that satisfy the properties of the definition of conditional expectation and it can be proved that $Z_1 = Z_2$ **a.s**.

Properties of Conditional Expectation

Theorem The following properties hold:

1.) $E(c|\mathcal{F}_0) = c$ **a.s.**

2.) $E(a_1.Y_1 + a_2.Y_2|\mathcal{F}_0) = E(a_1.Y_1|\mathcal{F}_0) + E(a_2.Y_2|\mathcal{F}_0)$ **a.s.** (i.e. it is *linear*.)

Example First property. Take $E(c|\mathcal{F}_0)$ and let $Z_1 = c$, then Z_1 satisfies the two properties of the definition of conditional expectation. Let $Z_2 = c + \mathbf{1}_A$ $A \in \mathcal{F}_0$ $P(A) = 0$. Z_2 also satisfies the two properties of the definition of conditional expectation.

Insert here Figure 1

Exercise Show that if $Z_1 = c$, then Z_1 satisfies the two properties of the definition of conditional expectation.

Proof 1.) c is $\mathcal{F}_0-$measurable. 2.) For every event $A \in \mathcal{F}_0 \to \int_A cdP = c = \int_A YdP$

3.) Conditional expectation is *monotone*:

$$If\ Y_1 \leq Y_2 \ \textbf{a.s} \text{ then } E(Y_1|\mathcal{F}_0) \leq E(Y_2|\mathcal{F}_0) \ \textbf{a.s}$$

4.) $|E(Y|\mathcal{F}_0)| \leq E(|Y| \ |\mathcal{F}_0)$ **a.s.**

5.) *Dominated Convergence Theorem*:

If $Y_{n_{n\to\infty}} \to Y$ **a.s** and there exists an integrable random variable Z s.t $|Y_n| \leq Z$, for every n, then

$$E(Y_n|\mathcal{F}_0)_{n\to\infty} \to E(Y|\mathcal{F}_0) \ \ \textbf{a.s}$$

and $E(Y|\mathcal{F})$ must be integrable. If we take Y_n integrable random variables and $Y_n \uparrow Y$ and Y is integrable by definition, we can use the dominated convergence threorem instead of monotone convergence theorem. It becomes redundant since in case of conditional expextation Y must be integrable.

Theorem $E(E(Y|\mathcal{F}_0)) = E(Y)$.

Proof We use the second property of conditional expectation

$$\begin{aligned} \int_A E(Y|\mathcal{F}_0)dP &= \int_A YdP \ \ \forall A \in \mathcal{F}_0 \\ \sin ce\ \Omega &\in \mathcal{F}_0 \text{ pick } A = \Omega \\ \int_\Omega E(Y|\mathcal{F}_0)dP &= \int_\Omega YdP \ \ \forall A \\ E(E(Y|\mathcal{F}_0)) &= E(Y). \text{ QED.} \end{aligned}$$

Theorem If Y is $\mathcal{F}_0$-measurable, then $E(Y|\mathcal{F}_0) = Y$ **a.s.**

Proof We have to prove that Y satisfies the definition of the conditional expectation.

1. Y is $\mathcal{F}_0$-measurable. (Y as Z in the definition).
2. $\int_A YdP = \int_A YdP$. (Y in LHS refers to RHS in the theorem, viceversa)

Proposition If Y is $\mathcal{F}_0$-measurable, then Y behaves like a constant. $E(Y|\mathcal{F}_0) = Y \Leftrightarrow E(c) = c$.

Theorem If Y is $\mathcal{F}_0$-measurable, then

$$E(Y.Z|\mathcal{F}_0) = Y.E(Z|\mathcal{F}_0) \ \textbf{a.s}$$

Proof Observe that Y behaves like a constant$\Rightarrow E(c.X) = c.E(X)$